

Riesz Integral Representation Theory

Abstract

This paper presents a Riesz integral representation theory in which functions, operators and measures take values in uniform commutative monoids (a commutative monoid with a uniformity making the binary operation of the monoid uniformly continuous). It describes the operators to which the theory can be applied and the finitely-additive measures they generate. For exactness, let S be a quasi-normal space (this includes all locally compact or normal spaces, and the products of connected such spaces), X and Z be uniform commutative monoids, \mathcal{F} a suitable family of functions on S to X , and ℓ an operator from \mathcal{F} to Z . The theory, which is applicable whenever S , X , \mathcal{B} , \mathcal{F} and \mathcal{T} generate a “Riesz system”, where \mathcal{B} is the family of (for example) all totally bounded subsets of X , yields necessary and sufficient conditions for ℓ to have a representation, $\ell(f) = \int f.d\nu_\ell$ for all $f \in \mathcal{F}$, as an integral with respect to a finitely additive measure, ν_ℓ . Operators satisfying the conditions will be called “Riesz integrals”. Given an underlying “Riesz system”, it is shown that every Riesz integral, ℓ , generates a certain kind of finitely additive measure, ν_ℓ , called here a “Riesz measure”. The correspondence between Riesz integrals and Riesz measures is a bijection. A straightforward calculation shows that if ℓ has such a representation, then it must have the Hammerstein property: $\ell(f + g_1 + g_2) + \ell(f) = \ell(f + g_1) + \ell(f + g_2)$, for all f , g_1 and g_2 in \mathcal{F} with g_1 and g_2 having “disjoint support”. When X and Z are topological vector spaces over the real or complex field, the theory yields necessary and sufficient conditions for operators with the Hammerstein property to be Riesz integrals. We note that uniform commutative monoids arise naturally when considering set-valued functions.

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1 Introduction

A quasi-uniform commutative monoid is a structure $(M, +, \mathcal{U})$ in which $(M, +)$ is a monoid, and \mathcal{U} is a quasi-uniformity [?] on M making $+$ quasi-uniformly continuous, that is, for all $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that if $(x, x'), (y, y') \in V$, then $(x + y, x' + y') \in U$. Riesz Integral Representation Theory gives conditions under which a map from a family of functions to a uniform commutative monoid (a commutative monoid with a uniformity under which the binary operation is uniformly continuous [?, ?]) is given by integration with respect to a finitely

additive measure [?, ?]. Although we need for the range of a measure only a quasi-uniform commutative monoid, it seems that the generation of a representing measure requires symmetry and satisfactory notions of completeness. For these reasons we shall consider uniformities in preference to quasi-uniformities. We note that uniform commutative monoids arise as hyperspaces when considering set-valued functions [?, ?, ?, ?, ?], an example being the family M_V^c of all closed subsets of a topological vector space, V , with the Hausdorff uniformity [?]. By considering the uniform commutative monoid thus obtained when $V = \mathbf{R}$, we see that our results are applicable to measures whose values are closed subsets of the real line.

In what follows (see Assumptions 2.3), S is a non-empty set, \mathcal{K} and \mathcal{G} are non-empty families of subsets of S , X and Z are uniform commutative monoids, \mathcal{B} is a non-empty family of subsets of X , \mathcal{F} is a family of X -valued functions on S , carrying a uniformity \mathcal{T} , under which it is a uniform commutative monoid, and ℓ is a map on \mathcal{F} to Z for which $\ell(0) = 0$. (The identity of a monoid will be denoted by 0. We will always assume that \mathcal{F} contains the function which is identically 0 on S , and is a commutative monoid under the binary operation induced by addition on X .) Intuitively, S corresponds to a normal or locally compact topological space, \mathcal{K} to the family of its closed subsets (respectively, closed compact subsets, when S is locally compact), and \mathcal{G} to its family of open subsets; \mathcal{B} corresponds to the family of closed, totally bounded subsets of the uniform commutative monoid X , \mathcal{F} to a “suitable” subfamily of the continuous X -valued functions on S with totally bounded range, and ℓ to a suitable function on \mathcal{F} to a uniform commutative monoid Z .

Riesz integral representation theory gives general conditions under which ℓ can be represented by an integral with respect to a Z^X -valued measure μ on a field containing $\mathcal{K} \cup \mathcal{G}$, that is,

$$\ell(f) = \int_S f.d\mu, \text{ for all } f \in \mathcal{F}.$$

For $x \in X$ and $y \in Z^X$, we will denote $y(x)$ by $x.y$. The integral will be a limit of finite sums, $\sum_{\alpha \in F} f(s_\alpha) \cdot \mu(\alpha)$, in which F is a finite, disjoint family of elements of the field, and $s_\alpha \in \alpha$ for each $\alpha \in F$. For the original result of F. Riesz [?], $S = [0, 1]$, $X = Z = \mathbf{R}$, \mathcal{F} is the family of continuous real-valued functions on $[0, 1]$, \mathcal{K} is the family of closed subsets of $[0, 1]$, and \mathcal{G} is the family of open subsets of $[0, 1]$.

Theorem 1.1 (F. Riesz, 1909) *ℓ is a continuous linear map on \mathcal{F} to Z if and only if there exists a Baire measure μ on the closed unit interval such that $\ell(f) = \int f d\mu$, for all $f \in \mathcal{F}$.*

This theorem has been extended to cover cases in which S is locally compact, Z is any locally convex topological vector space, \mathcal{F} is a subfamily of the X -valued continuous functions on S with totally bounded range [Assumptions 2.3, Remark 2.5.1], with a uniformity \mathcal{T} for which $\mathcal{T} \subseteq \mathcal{U}_c$, and ℓ is a continuous, linear Z -valued map on \mathcal{F} [?, ?]. The theory presented by B. Mair in [?] covers most of the then known theorems for locally compact S , locally convex X and Z , and

continuous linear ℓ . Normal topological spaces S have been considered only in [?]. Non-linear operators have been discussed in [?, ?, ?]. An approach using dominated operators [?] was introduced in [?]. The problem continues to receive attention [?, ?, ?, ?, ?, ?, ?, ?]. Necessary and sufficient conditions are given in [?] for μ to be scalar valued when S is compact, $X = Z$ is a Banach space, and \mathcal{F} is the space of continuous X -valued functions on S under the sup-norm. Non-topological structures have been introduced in [?, ?]. Applications to the theory of weakly compact operators are considered in [?, ?]. We will develop a general theory which provides a unified setting for the foregoing, is applicable whether S is locally compact or normal (quasi-normal spaces, Remark 2.5.2), allows X , Z and \mathcal{F} to be uniform commutative monoids, and yields integral representations even for non-linear operators. Our approach is as follows. For suitable ℓ ,

- (a) we shall generate a finitely additive, Z^X -valued set function τ_ℓ on \mathcal{K} .
- (b) This set function is extended to a regular, finitely additive measure ν_ℓ on a certain field containing $\mathcal{K} \cup \mathcal{G}$.
- (c) It is verified that $\ell(f) = \int f.d\nu_\ell$ for all $f \in \mathcal{F}$, and that the mapping $\ell \rightarrow \nu_\ell$ is a bijection.

The σ -additivity of ν_ℓ may be guaranteed by compactness conditions on \mathcal{K} [?], or by conditions on ℓ . Since ν_ℓ is *regular* with respect to $(\mathcal{K}, \mathcal{G})$ (Theorem 4.12), then ν_ℓ is always σ -additive when \mathcal{K} is the family of compact subsets of a Hausdorff, locally compact space S , and \mathcal{G} its family of open subsets [?].

The generation of a measure ν_ℓ from ℓ is the most technical part of our discussion. The present process modifies the approximation process for positive measures. Given the function τ_ℓ induced on \mathcal{K} by ℓ , we define functions ξ_ℓ on \mathcal{G} and ν_ℓ on the subsets of S by

$$\xi_\ell(\gamma) = \lim_{\kappa \in \mathcal{K}, \kappa \subseteq \gamma} \tau_\ell(\kappa), \quad \nu_\ell(\alpha) = \lim_{\gamma \in \mathcal{G}, \alpha \subseteq \gamma} \xi_\ell(\gamma).$$

Under the conditions given, the limits always exist (Proposition 4.9.7).

The construction of ν_ℓ from ℓ is based on three notions, **Riesz system**, **Riesz integral** over a Riesz system, and **Riesz measure**. Their definitions abstract the properties used to ensure that this construction yields a bijection $\ell \leftrightarrow \nu_\ell$ (Theorem 5.1).

Let \mathcal{E} be the smallest field of subsets of S containing $\mathcal{K} \cup \mathcal{G}$. Functions $g_1, g_2 \in \mathcal{F}$ will be called **\mathcal{E} -separated** if and only if there exist disjoint $E_1, E_2 \in \mathcal{E}$ such that $g_i(x) = 0$ for all $x \in S \setminus E_i$, $i = 1, 2$. The operator ℓ has the **Hammerstein property** relative to \mathcal{E} [?] if and only if $\ell(f + g_1 + g_2) + \ell(f) = \ell(f + g_1) + \ell(f + g_2)$, for all \mathcal{E} -separated g_1 and g_2 in \mathcal{F} . If ℓ has an integral representation as described above, then it must have the Hammerstein property. Indeed, consider $f, g_1, g_2 \in \mathcal{F}$, with E_1, E_2 being disjoint members of \mathcal{E} such that $g_i(x) = 0$ for all $x \in S \setminus E_i$, $i = 1, 2$. Since integration yields, for each function $f \in \mathcal{F}$, an additive set function, $E \in \mathcal{E} \rightarrow \int_E f d\mu \in Z$ [?], then

$$\begin{aligned}
& \ell(f + g_1) + \ell(f + g_2) = \int_S (f + g_1) d\mu + \int_S (f + g_2) d\mu \\
&= \int_{S \setminus E_1} (f + g_1) d\mu + \int_{E_1} (f + g_1) d\mu + \int_{S \setminus E_2} (f + g_2) d\mu + \int_{E_2} (f + g_2) d\mu \\
&= \int_{S \setminus E_1} f d\mu + \int_{E_1} (f + g_1 + g_2) d\mu + \int_{S \setminus E_2} f d\mu + \int_{E_2} (f + g_1 + g_2) d\mu \\
&= \int_{S \setminus (E_1 \cup E_2)} f d\mu + \int_{E_2} f d\mu + \int_{E_1} (f + g_1 + g_2) d\mu + \\
&\quad \int_{S \setminus (E_1 \cup E_2)} (f + g_1 + g_2) d\mu + \int_{E_1} f d\mu + \int_{E_2} (f + g_1 + g_2) d\mu \\
&= \ell(f) + \ell(f + g_1 + g_2).
\end{aligned}$$

Theorem 5.1 establishes a one-to-one correspondence between Riesz integrals and Riesz measures. When X , Z and \mathcal{F} are topological vector spaces, Example 3.6.6 establishes sufficient conditions for an operator with the Hammerstein property to be a Riesz integral.

References of this paper will be given in one of the formats *type:section:subsection* or *type:section:subsection:number*, where “type” may be any one of *Assumption*, *Remark*, *Definition*, *Theorem*, *Example*, *Notation*, or their plurals.

Throughout the sequel, \mathbf{N} denotes the set of whole numbers. Let $0 = \emptyset$, and for each $n \in \mathbf{N}$, let $n = \{0, 1, \dots, n-1\}$. For any set X , a **sequence** in X is a function on \mathbf{N} to X . For any function f , and argument α , we denote the set $\{f(x) : x \in \alpha\}$ by $f^\wedge \alpha$, and $f(\alpha, x)$ by $(f(\alpha))(x)$. For each family \mathcal{H} of sets, $\bigcup \mathcal{H}$ denotes the union of all elements of \mathcal{H} , and $\bigcap \mathcal{H}$ their intersection. For all $\alpha \subseteq \bigcup \mathcal{H}$, we define $\mathcal{H}\text{-hull}(\alpha)$ to be $\bigcap \{\eta \in \mathcal{H} : \alpha \subseteq \eta\}$, where $\bigcap \emptyset = \bigcup \mathcal{H}$. When $\alpha = \{x\}$, we shall write $\mathcal{H}\text{-hull}(x)$ for $\mathcal{H}\text{-hull}(\{x\})$. We say that \mathcal{H} is **closed under finite intersections (closed under finite unions)** if and only if $\bigcap \mathcal{H}' \in \mathcal{H}$ ($\bigcup \mathcal{H}' \in \mathcal{H}$) for all finite $\mathcal{H}' \subseteq \mathcal{H}$. (Thus, in the former case, $\bigcap \emptyset = \bigcup \mathcal{H} \in \mathcal{H}$, and in the latter, $\bigcup \emptyset = \emptyset \in \mathcal{H}$.)

For basic information on topologies, uniformities and quasi-uniformities, nets and filters, we refer to [?, ?, ?, ?, ?]. *Topological spaces will be always Hausdorff*. The **closure** of a subset α of a topological space S will be denoted by α^{cl} . The **support** of a function f , on a topological space S to a set X with a distinguished element 0 , is $S \setminus \bigcup \{\gamma : \gamma \subseteq S \text{ is open, and } f^\wedge \gamma = 0\}$. (This evidently generalizes, to an arbitrary S , the notion of “support” when S is locally compact.) We shall say that a function f has *compact support* if the support of f is compact. Let (X, \mathcal{U}) be a uniform space. For all $x \in X$ and $U \in \mathcal{U}$, we denote $\{t : (x, t) \in U\}$ by U_x . A set $E \subseteq X$ is **totally bounded** if and only if for each $U \in \mathcal{U}$ there exists a finite $F \subseteq E$ such that $E \subseteq \bigcup_{x \in F} U_x$.

For any subset V of a Cartesian product $X \times X$, V^{-1} denotes the set of all (x, y) such that $(y, x) \in V$. The set V is said to be **symmetric** if $V = V^{-1}$. Let $(X, +)$ be a commutative monoid. A subset U of $X \times X$ is **translation invariant** if and only if $(x + t, y + t) \in U$ for all $(x, y) \in U$ and $t \in X$. A **uniform commutative monoid** is a structure $(X, +, \mathcal{U})$ such that $(X, +)$ is a commutative monoid, and \mathcal{U} is a filter of $X \times X$ such that for all $U \in \mathcal{U}$ (i) U contains the diagonal of X , $\{(x, x) : x \in X\}$, and there exists $V \in \mathcal{U}$ such that (ii) $V \circ V \subseteq U$ (iii) if $(x, x'), (y, y') \in V$ then $(x + y, x' + y') \in U$. For each uniform commutative monoid $(X, +, \mathcal{U})$, denote by $unif X$ the base for

\mathcal{U} consisting of its closed (in the product topology induced by the uniformity), translation invariant, symmetric sets [?]. If A and B are subsets of X , we denote the set $\{x + y : x \in A, y \in B\}$ by $A + B$. The real and complex fields will always carry the uniformity generated by the metric $d(x, y) = \|x - y\|$.

A function f on a directed set (D, \prec) into a uniform space (X, \mathcal{U}) is a **Cauchy net** if and only if for each $U \in \mathcal{U}$, there exists $i \in D$ such that $(f(j), f(k)) \in U$ for all j, k with $i \prec j$ and $i \prec k$. A filter base F in (X, \mathcal{U}) is a **Cauchy filter base** if and only if for each $U \in \mathcal{U}$, there exists $\alpha \in F$ such that $\alpha \times \alpha \subseteq U$. When D is the family of finite subsets of some set I directed by inclusion, and X is a uniform commutative monoid, we say that f is **partial-sum Cauchy** if and only if for each $U \in \mathcal{U}$, there exists $i \in D$ such that $(0, f(k)) \in U$ for all k with $i \cap k = \emptyset$. When X is a uniform commutative monoid, every net which is partial sum Cauchy is Cauchy, and when X is a topological group then every Cauchy net of finite partial sums is partial-sum Cauchy. A net f in (X, \mathcal{U}) is said to be a **null net** iff for each $U \in \mathcal{U}$ there exists $i \in D$ such that $(0, f(j)) \in U$ for all $i \prec j$. An X -valued function f on a set I is **quasi-summable** if and only if the net

$$\left\{ \sum_{x \in J} f(x) : J \text{ is a finite subset of } I \right\}$$

is a Cauchy net in X . A subset E of a uniform commutative monoid X will be called **quasi-perfect** if and only if a function f is quasi-summable whenever its family of finite partial sums is contained in E . A subset E of a uniform commutative monoid X will be called **perfect** if and only if a function f is summable [?, ?] whenever its family of finite partial sums is contained in E . Thus, if a set is quasi-perfect and relatively complete, then it is necessarily perfect.

For any topological space S , and uniform, commutative monoid X , we denote by $\mathcal{C}_c(S, X)$ the space of (uniformly) continuous X -valued functions on S with compact support, by $\mathcal{C}_p(S, X)$ the space of uniformly continuous X -valued functions on S with totally bounded range, and by $\mathcal{C}(S, X)$ the space of all uniformly continuous X -valued functions on S . If \mathcal{K} is a family of subsets of S , we denote by $\mathcal{C}_{\mathcal{K}}(S, X)$ the family of all uniformly continuous X -valued functions on S with totally bounded range, and support contained in some $K \in \mathcal{K}$; when X is a field of scalars, \mathbf{R} or \mathbf{C} , explicit mention of it will usually be omitted. In each of the foregoing cases, the space carries the uniformity of uniform convergence on S , unless stated otherwise. When X admits multiplication by $\mathcal{C}_p(S)$, then $X \otimes \mathcal{C}_{\mathcal{K}}(S)$ denotes the space in $\mathcal{C}(S, X)$ spanned by functions of the form xf , $x \in X$ and $f \in \mathcal{C}_{\mathcal{K}}(S)$. When Ω is an open subset of \mathbf{R}^n we denote by $\mathcal{C}_c^\infty(\Omega, \mathbf{R}^m)$ the space of infinitely-differentiable, \mathbf{R}^m -valued functions with compact support, [?], p. 287, with the uniformity of uniform convergence in all derivatives on compacta. When X and Z are both topological vector spaces, we denote by $L_{\mathcal{B}}(X, Z)$ the space of continuous linear maps from X to Z with the uniformity of uniform convergence on the members of \mathcal{B} (assumed to be directed by \subseteq , and such that the image of each B is bounded), and by $L_\sigma(X, Z)$

the space of continuous linear maps from X to Z with the uniformity of uniform convergence on the finite subsets of X (point-wise convergence) [?].

A topological space S will be called **quasi-normal** under $(\mathcal{K}, \mathcal{G})$ if \mathcal{K} is a subfamily of its closed subsets, and \mathcal{G} is a subfamily of its open subsets, such that \mathcal{K} is closed under finite unions, \mathcal{G} is closed under finite intersections and finite unions, and for all $\kappa \in \mathcal{K}$ and $\gamma \in \mathcal{G}$: (i) $\kappa \setminus \gamma \in \mathcal{K}$, $\gamma \setminus \kappa \in \mathcal{G}$, (ii) if $\kappa \subseteq \gamma$ then there exist $\kappa' \in \mathcal{K}$ and $\gamma' \in \mathcal{G}$ such that $\kappa \subseteq \gamma' \subseteq \kappa' \subseteq \gamma$. A topological space S will be called *quasi-normal* if and only if it is quasi-normal under $(\mathcal{K}', \mathcal{G}')$ for some \mathcal{K}' and \mathcal{G}' . S will be called **semi-connected quasi-normal** under $(\mathcal{K}, \mathcal{G})$ if and only if it is quasi-normal under $(\mathcal{K}, \mathcal{G})$ and $K \not\subseteq G_1 \cup G_2$ for any $K \in \mathcal{K}$, and non-empty, disjoint $G_1, G_2 \in \mathcal{G}$. S is *semi-connected quasi-normal* iff S is semi-connected and quasi-normal under $(\mathcal{K}, \mathcal{G})$, for some \mathcal{K} and \mathcal{G} . Clearly, every locally compact or normal space is quasi-normal [?, ?], and every metrisable, topological vector space over the real or complex field is semi-connected and quasi-normal. We note that the arbitrary product of semi-connected quasi-normal topological vector spaces is again quasi-normal [?]. In particular, the arbitrary product of metrisable, topological vector spaces is quasi-normal, but not necessarily metrisable [?, ?].

A **measure** [?] is a set function h with values in a commutative monoid such that (i) $\emptyset \in \text{dom } h$ and $h(\emptyset) = 0$, for all $A, B \in \text{dom } h$, (ii) $A \cap B \in \text{dom } h$, $A \cup B \in \text{dom } h$ and (iii) $h(A \cup B) + h(A \cap B) = h(A) + h(B)$. When h takes values in a *uniform* commutative monoid, then h is **countably additive** if and only if h is a measure, and $h(\bigcup A) = \sum_{n \in \mathbf{N}} h(A_n)$ for every disjoint sequence A in $\text{dom } h$. Riesz representation theory for Banach space-valued functions is discussed in [?], pp.59,84,151.

We stress that the general theory covers all of the topological results mentioned in the papers cited above. In particular, it provides a common theory for locally compact and normal spaces S – to the authors’s knowledge, normal spaces are considered only in the paper of [?]. Further, the range of ℓ may now be any topological vector space in which every bounded subset is perfect and relatively complete. Thus, for stochastic processes, $\mathcal{C}_p(S, X)$, on a space S , with S being quasi-normal, X, Z being topological vector spaces, the general theory (Remark 2.5.2, Theorem 5.1) yields a representation not provided by any combination of the cited papers:

Theorem 1.2 *Let S be quasi-normal under $(\mathcal{K}, \mathcal{G})$, X be a topological vector space, and $Z = L_0(\lambda)$. Then, ℓ is a continuous linear map from $\mathcal{C}_p(S, X)$ to Z if and only if $\ell(f) = \int f d\nu_\ell$ for some unique finitely-additive, $L_\sigma(X, Z)$ -valued Riesz measure, ν_ℓ , on a field containing $\mathcal{K} \cup \mathcal{G}$.*

Further, we have

Theorem 1.3 *Let ℓ be a continuous linear map from $\mathcal{C}_p(\Omega, R^m)$ to Z . If ℓ is continuous with respect to the topology of uniform convergence on Ω , then ℓ has an integral representation*

$$\ell : f \in \mathcal{C}_p(\Omega, R^m) \rightarrow \int f d\nu_\ell$$

Certainly, we have that “uniform commutative monoids” \subseteq “linear monoids”

\subseteq “topological vector spaces”. In particular, therefore we may consider take X and Z to be topological vector spaces, in view Remark 2.5.1. Note that distributivity over scalar addition is the axiom needed to make a linear monoid into a topological vector spaces. However, this property seems to be needed in the proof that every quasi-normal space generates a Riesz system.

We close this section with the observation that the family of closed subsets of any uniform commutative monoid M itself becomes a uniform commutative monoid, under the uniformity having as a base all sets of the form $\{(A, B) : \forall x \in A \exists y \in B (x, y) \in U, \forall y \in B \exists x \in A (x, y) \in U\}$, for some $U \in \text{unif} M$. Thus the study of functions and measures with values in the family of closed subsets of a uniform commutative monoid leads to consideration of monoid-valued functions and measures [?, ?, ?].

2 Riesz Systems

Throughout the sequel we shall adhere to the notation of the introduction, and use the informal viewpoint suggested there as motivation for the following definitions and assumptions.

Definitions 2.1 Let $\alpha \subseteq S$, and $x \in X$. A function f on S to X is **supported by** α (denoted by $f \prec \alpha$) if and only if there exists $\kappa \in \mathcal{K}$ such that $\kappa \subseteq \alpha$ and $f(s) = 0$ for all $s \in S \setminus \kappa$; f **equals x over** α (denoted by $\alpha =_x f$) if and only if there exists $\gamma \in \mathcal{G}$ with $\alpha \subseteq \gamma$ such that $f(s) = x$ for all $s \in \gamma$.

Notation 2.2

$\mathcal{B}_0 = \bigcap \{\mathcal{H} \subseteq \mathcal{B} : \mathcal{H} \text{ is closed under arbitrary intersections, } \bigcap \mathcal{H} = 0\}$,
 $\mathcal{B}\text{-hull}(x) \in \mathcal{H}$ for all $x \in X$, $\mathcal{B}\text{-hull}(H_1 + H_2) \in \mathcal{H}$ for all $H_1, H_2 \in \mathcal{H}$;
 $\mathcal{F}_A := \{f \in \mathcal{F} : \text{rng} f \subseteq A\}$, for each $A \subseteq X$;
 $\mathcal{F}_0 := \{f \in \mathcal{F} : \text{rng} f \subseteq B \text{ for some } B \in \mathcal{B}_0\}$.

$(S, (\mathcal{K}, G), (X, \mathcal{B}), (\mathcal{F}, T))$, denoted by \mathfrak{R} , is called a **Riesz system**, if and only if S is set, \mathcal{K}, \mathcal{G} are families of subsets of S , X is a uniform commutative monoid, \mathcal{B} is a family of subsets of X , \mathcal{F} is a family of X -valued functions on S , and T is a uniformity on \mathcal{F} , such that the following assumptions hold.

Assumptions 2.3

On \mathcal{K}, G :

- (1) \mathcal{K} is closed under finite unions;
- (2) \mathcal{G} is closed under finite intersections and finite unions;
- (3) for all $\kappa \in \mathcal{K}$ and $\gamma \in \mathcal{G}$, $\kappa \setminus \gamma \in \mathcal{K}$ and $\gamma \setminus \kappa \in \mathcal{G}$;
- (4) for all $\kappa \in \mathcal{K}$ and $\gamma \in \mathcal{G}$ with $\kappa \subseteq \gamma$ there exist $\gamma' \in \mathcal{G}$ and $\kappa' \in \mathcal{K}$ with $\kappa \subseteq \gamma' \subseteq \kappa' \subseteq \gamma$.

(A similar idea is used by M. Sion and A. Sapounakis [?], and also by Panchapagesan [?, ?]. Note that the assumptions above lead to the following separation property: for all disjoint κ_1, κ_2 in \mathcal{K} there exist disjoint $\gamma_1, \gamma_2 \subseteq \mathcal{G}$ such that $\kappa_i \subseteq \gamma_i$, $i = 1, 2$.)

On \mathcal{B} :

- (5) \mathcal{B} is closed under arbitrary, non-empty intersections, and $0 \in \bigcap \mathcal{B}$;

- (6) for all $B, B' \in \mathcal{B}$ there exists $C \in \mathcal{B}$ such that $B + B' \subseteq C$;
- (7) for each $x \in X$ there exists $B \in \mathcal{B}$ with $x \in B$.

On \mathcal{F} :

- (8) \mathcal{F} contains the function which is identically 0 on S , and, under the addition $+$ induced by X , is a uniform commutative monoid with respect to the uniformity \mathcal{T} ;
- (9) for each $f \in \mathcal{F}$ there exists $B \in \mathcal{B}$ with $\text{rng } f \subseteq B$;
- (10) for each $f \in \mathcal{F}$ and $W \in \text{unif } Z$ there exists a finite $G \subseteq \mathcal{G}$ such that $S = \bigcup G$ and for all $\gamma \in G$ and $s, t \in \gamma$, $(f(s), f(t)) \in W$ (we say that f is *finitely \mathcal{G} -partitionable* — see [?] for the definition of *partitionability*);
- (11) For all $\kappa \in \mathcal{K}, \gamma \in \mathcal{G}$ with $\kappa \subseteq \gamma$, and $x \in X$, there exists $f \in \mathcal{F}$ such that $\text{rng } f \subseteq \mathcal{B}\text{-hull}(x)$, f is supported by γ , and f equals x over κ . (This assumption ensures that \mathcal{F} contains enough functions to approximate constant functions on members of \mathcal{K} .)
- (12) For all $B \in \mathcal{B}$ and $T \in \mathcal{T}$, there exists $U \in \text{unif } X$ such that for all $\kappa \in \mathcal{K}, \gamma \in \mathcal{G}$ with $\kappa \subseteq \gamma$, and $f, g \in \mathcal{F}_B$: if f and g are both supported by γ , and for some $\omega \in \mathcal{G}$ with $\kappa \subseteq \omega \subseteq \gamma$ we have that $(f(s), g(s)) \in U$ for all $s \in \omega$, then there exist $p, q \in \mathcal{F}_B$ such that p and q are both supported by $\gamma \setminus \kappa$, and $(f + p, g + q) \in T$. (This relationship will be denoted by $T \text{ ext}_B U$. The assumption says that if f, g are U -close, in the manner specified, then they have extensions which are T -close everywhere.)
- (13) For all $B \in \mathcal{B}$, finite $G \subseteq \mathcal{G}$, $\kappa \in \mathcal{K}$ with $\kappa \subseteq \bigcup G$, and $f \in \mathcal{F}_B$, there exists a function g on G to \mathcal{F}_B such that for all $\gamma \in G$: g_γ is supported by γ , $\sum_{\gamma \in J} g_\gamma \in \mathcal{F}_B$ for all $J \subseteq G$, and $f(s) = \sum_{\gamma \in G} g_\gamma(s)$ for all $s \in \kappa$. (Thus, for each $\kappa \in \mathcal{K}$, there is a “partition of unity” on κ .)

By (4) and (11), for all $B \in \mathcal{B}$ and non-empty $\gamma \in \mathcal{G}$, there exists $f \in \mathcal{F}_B, f \neq 0$, such that f is supported by γ . If $(S, (\mathcal{K}, G), (X, \mathcal{B}), (\mathcal{F}, T))$ is a Riesz system, with X being a topological vector space, then it is easily checked that $(S, (\mathcal{K}, G), (X, \mathcal{B}_0), (\mathcal{F}_0, T))$ is also a Riesz system.

Proof. For suppose that $(S, (\mathcal{K}, G), (X, \mathcal{B}), (\mathcal{F}, T))$ is a Riesz system. Then \mathcal{K}, \mathcal{G} trivially satisfy Assumptions (1) – (4). By the conditions on \mathcal{B} , since, in particular, $B_i \subseteq \mathcal{B}\text{-hull}(B_1 + B_2) \in \mathcal{B}_0$ for $i = 1, 2$, then \mathcal{B}_0 satisfies Assumption (6). Thus \mathcal{B}_0 satisfies Assumptions (5) – (7). Clearly $\mathcal{B}\text{-hull}(0) \in \mathcal{B}_0$, therefore Assumption (8) is satisfied by \mathcal{F}_0 . Clearly, by the definition of \mathcal{F}_0 , it must satisfy (9) – (13). \square

Further, when \mathcal{B} is the family of closed, balanced, totally bounded subsets of a topological vector space X , then \mathcal{B}_0 is a subfamily of the family of all closed, balanced, bounded subsets of finite dimensional subspaces of X .

Examples 2.4

In the following examples, X is any topological vector space [?].

- .1 S is a set, \mathcal{R} is an algebra of subsets of S , and $\mathcal{K} = \mathcal{G} = \mathcal{R}$; \mathcal{B} is the family of all closed, balanced totally bounded subsets of X ; \mathcal{F} is the family of totally measurable X -valued functions [?, ?], that is, the uniform closure in X^S of the family of all simple functions $\sum_{\rho \in R} x_\rho \chi_\rho$, (where R is a finite,

disjoint subfamily of \mathcal{R} with $S = \bigcup R$, x is a function on R to X , and for each $\rho \in R$, χ_ρ denotes the characteristic function of ρ ,) and \mathcal{T} is the uniformity of uniform convergence on S . (Riesz integral representation of linear operators on $M(S, X)$, theorem 5.1, leads to the Fichtenholz-Hildebrandt-Kantorovitch theorem [?].)

- .2 S is a locally compact space, \mathcal{K}, \mathcal{G} are respectively its family of compact subsets, and its family of open subsets; \mathcal{B} is the family of closed, balanced, totally bounded subsets of X ; $\mathcal{F} = \mathcal{C}_c(S, X)$, the space of continuous X -valued functions on S with compact support, and \mathcal{T} is the uniformity of uniform convergence on S .
- .3 S is a normal space, \mathcal{K}, \mathcal{G} are respectively its family of closed subsets, and its family of open subsets; \mathcal{B} is the family of closed, balanced, totally bounded subsets of X ; $\mathcal{F} = \mathcal{C}_p(S, X)$, the space of totally bounded, continuous X -valued functions on S , and \mathcal{T} is the uniformity of uniform convergence on S .

For Example 2.4.1 it is readily verified that the given elements constitute a Riesz system. For Examples 2.4.2 and 2.4.3 the verifications are more technical. In each of these two cases the pair $(\mathcal{K}, \mathcal{G})$ satisfies the assumptions given. Also, \mathcal{B} satisfies Assumptions (5) – (7) [?], and for each $f \in \mathcal{F}$, $\text{rng } f \subseteq \mathcal{B}$ for some $B \in \mathcal{B}$. Further, $(\mathcal{F}, +)$ is a topological vector space under the uniformity \mathcal{T} . Thus Assumptions (1) – (9) are satisfied. We show below that the remaining assumptions on \mathcal{F} are valid.

- (10) If $f \in \mathcal{F}$ then it has totally bounded range. Thus, for each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ with $V \circ V \subseteq U$, and finite $F \subseteq X$ such that $\text{rng } f \subseteq \bigcup_{x \in F} V_x$. For each $x \in F$ let $\gamma_x = f^{-1}(V_x)$. Then $\gamma_x \in \mathcal{G}$ and $(f(s), f(t)) \in U$ for all s, t in γ_x . \square

For (11) – (13) we use the following fact, true in Examples 2 and 3: given any $\kappa \in \mathcal{K}$ and $\gamma \in \mathcal{G}$ with $\kappa \subseteq \gamma$ there exists a continuous function on S to $[0, 1]$, identically 1 on κ and vanishing on $S \setminus \gamma$, [?, ?]. For the corresponding result on \mathcal{C}_c^∞ functions, $f : \Omega \rightarrow \mathbf{R}^m$, where Ω is an open subset of \mathbf{R}^n , and f has compact support, see [?], p.25, or [?], p.385. Thus, we have that $(\Omega, (\mathcal{K}, \mathcal{G}), (\mathbf{R}^m, \mathcal{B}), (\mathcal{C}_c^\infty(\Omega, \mathbf{R}^m), \mathcal{V}))$ is a Riesz system, when \mathcal{V} is the uniformity of uniform convergence on S . However, it is not a Riesz system when \mathcal{V} is the uniformity of uniform convergence on compacta in all derivatives.

- (11) Let $x \in X, \kappa \in \mathcal{K}$ and $\gamma \in \mathcal{G}$ with $\kappa \subseteq \gamma$. By the foregoing remark and Assumption (4) there exist $\kappa_1, \kappa_2 \in \mathcal{K}$ and $\gamma_1, \gamma_2 \in \mathcal{G}$, such that $\kappa \subseteq \gamma_1 \subseteq \kappa_1 \subseteq \gamma_2 \subseteq \kappa_2 \subseteq \gamma$ and a continuous function h on S to $[0, 1]$ which is 1 on κ_1 and vanishes on $S \setminus \gamma_1$. Let $f(s) = x \cdot h(s)$ for all $s \in S$. \square
- (12) Let $B \in \mathcal{B}, U, V \in \mathcal{U}$ be balanced, with $U \circ U \subseteq V$, $\kappa \in \mathcal{K}, \gamma \in \mathcal{G}, \omega \in \mathcal{G}$, with $\kappa \subseteq \omega \subseteq \gamma$, and $f, g \in \mathcal{F}_B$, both supported by γ , be such that $(f(s), g(s)) \in U$ for all $s \in \omega$. By Assumption (4) and the preceding remark on the existence of continuous functions on S , there exist $\kappa' \in$

$\mathcal{K}, \gamma' \in \mathcal{G}$, and a continuous function h on S to $[0,1]$, such that $\kappa \subseteq \gamma' \subseteq \kappa' \subseteq \omega$, and h is 1 on κ' , and vanishes on $S \setminus \omega$. Let $p = (1-h)g$ and $q = (1-h)f$. Then p and q are both in \mathcal{F}_B , and $(p(s), q(s)) \in U$ for all $s \in \omega$. Further, since f and g are supported by γ , then there exist κ_1, κ_2 in \mathcal{K} such that $\kappa \subseteq \kappa_i \subseteq \gamma$ for $i = 1, 2$, $f(s) = 0$ for all $s \notin \kappa_1$, and $g(s) = 0$ for $s \notin \kappa_2$. Then $p(s) = q(s) = 0$ for all $s \notin (\kappa_1 \cup \kappa_2) \setminus \gamma'$, and $(\kappa_1 \cup \kappa_2) \setminus \gamma' \subseteq \gamma \setminus \kappa$. Thus p, q are both supported by $\gamma \setminus \kappa$, and for all $s \in S$: $((f+p)(s), (g+q)(s)) \in V$, since

$$\begin{aligned} (f(s), g(s)) &\in U && \text{if } s \in \kappa' \\ ((f+p)(s), (g+q)(s)) &\in U \circ U && \text{if } s \in \omega \setminus \kappa', \\ ((f+g)(s), (g+f)(s)) &\in U, && \text{for all } s \in S \setminus \omega. \quad \square \end{aligned}$$

- (13) Let $\kappa_0 \in \mathcal{K}$, $B \in \mathcal{B}$, $f \in \mathcal{F}_B$ and, for some $n \in \mathbf{N}$, let $\{G_0, \dots, G_{n-1}\} \subseteq \mathcal{G}$ be such that $\kappa_0 \subseteq G_0 \cup \dots \cup G_{n-1}$. Let $\alpha_0 = \kappa_0 \setminus \bigcup_{1 \leq j \leq n-1} G_j$. Then $\alpha_0 \in \mathcal{K}$ and $\alpha_0 \subseteq G_0$. Hence there exist $\gamma_0, \beta_0 \in \mathcal{G}$ and a continuous function ϕ_0 on S to $[0,1]$, such that $\bar{\gamma}_0, \bar{\beta}_0 \in \mathcal{K}$, $\alpha_0 \subseteq \gamma_0 \subseteq \bar{\gamma}_0 \subseteq \beta_0 \subseteq \bar{\beta}_0 \subseteq G_0$, $\phi_0(s) = 1$ on $\bar{\gamma}_0$, and $\phi_0(s) = 0$ on $S \setminus \beta_0$. Let $\kappa_1 = \kappa_0 \setminus \gamma_0$. Then $\kappa_1 \in \mathcal{K}$ and $\kappa_1 \subseteq \bigcup_{1 \leq j \leq n-1} G_j$. Clearly, $\kappa_0 \subseteq \gamma_0 \cup \kappa_1$. Repeating the argument for each i with $1 \leq i \leq n-1$, we find sets $\alpha_i, \kappa_i \in \mathcal{K}$, $\beta_i, \gamma_i \in \mathcal{G}$, such that

$$\begin{aligned} \bar{\beta}_i, \bar{\gamma}_i &\in \mathcal{K}, \\ \kappa_i &= \kappa_{i-1} \setminus \gamma_{i-1} \text{ and } \kappa_i \subseteq \bigcup_{i \leq j \leq n-1} G_j, \\ \alpha_i &= \kappa_i \setminus \bigcup_{i+1 \leq j \leq n-1} G_j, \\ \alpha_i &\subseteq \gamma_i \subseteq \bar{\gamma}_i \subseteq \beta_i \subseteq \bar{\beta}_i \subseteq G_i, \end{aligned}$$

and a continuous function ϕ_i on S to $[0,1]$ such that $\phi_i(s) = 1$ on $\bar{\gamma}_i$ and $\phi_i = 0$ on $S \setminus \beta_i$. Let $\omega = \gamma_0 \cup \dots \cup \gamma_{n-1}$. Then $\kappa_0 \subseteq \omega \subseteq \bigcup_{0 \leq j \leq n-1} G_j$. Let ϕ_n be a continuous function on S to $[0,1]$ such that ϕ_n is 0 on κ and 1 on $S \setminus \omega$. Then $\sum_{j \leq n} \phi_j(s) > 0$ for all $s \in S$. Let

$$g_j = \phi_j f / \sum_{j \leq n} \phi_j, \quad 0 \leq j \leq n-1.$$

Then g_j is supported by G_j , $\sum_{j \in J} g_j \in \mathcal{F}_B$ for all $J \subseteq \{0, \dots, n-1\}$, and $f(s) = \sum_{j \leq n-1} g_j(s)$ for all $s \in \kappa_0$. \square

Remarks 2.5

- .1 In Examples 2.4.2 and 2.4.3, we may take \mathcal{F} to be any subset of $\mathcal{C}_p(S, X)$, satisfying Assumptions (8) – (10), which is a unital module over $\mathcal{C}_p(S)$ containing $X \otimes \mathcal{C}_K(S)$, and \mathcal{T} to be any uniformity, coarser than that of uniform convergence on S , under which \mathcal{F} is a uniform, commutative monoid, such as the uniformity of uniform convergence on countable subsets of S ; or the uniformity generated by the countable-open topology in [?]. In particular, we may take \mathcal{T} to be the uniformity of uniform convergence on any family of subsets of S directed by \subseteq . By taking \mathcal{T} to be a uniformity coarser than \mathcal{U}_c , we ensure that the proof of Assumption (12) for Examples 2.4.2 and 2.4.3 holds for \mathcal{T} .

.2 If S is quasi-normal under $(\mathcal{K}, \mathcal{G})$, then, by repeating the proof of Urysohn's lemma [?, ?], we can show that, given any $\kappa \in \mathcal{K}$ and $\gamma \in \mathcal{G}$ with $\kappa \subseteq \gamma$ there exists a continuous function on S to $[0,1]$, identically 1 on κ and vanishing on $S \setminus \gamma$. Following the verifications of Examples 2.4.2 and 2.4.3, it can now be shown that $(S, (\mathcal{K}, \mathcal{G}), (X, \mathcal{B}), (\mathcal{F}, \mathcal{T}))$ is a Riesz system when S is quasi-normal under $(\mathcal{K}, \mathcal{G})$, X is a topological vector space, \mathcal{B} is the family of closed, totally bounded subsets of X , \mathcal{F} is a subset of $\mathcal{C}_p(S, X)$ satisfying Assumptions (8) – (10, which is closed under multiplication by functions in $\mathcal{C}_p(S)$, and contains $X \otimes \mathcal{C}_\mathcal{K}(S)$, and \mathcal{T} is a uniformity on \mathcal{F} , coarser than the uniformity of uniform convergence on S , under which \mathcal{F} is a uniform commutative monoid. Riesz systems can therefore be constructed for any quasi-normal space.

3 Integrals

Intuitively, the map ℓ is an *integral* if and only if it is given by integration, $f \in \mathcal{F} \rightarrow \int f.d\nu_\ell$, with respect to some finitely-additive, Z^X -valued measure ν_ℓ on S . We shall give the construction of ν_ℓ in the following section. However, the description of the class of operators to be considered is reasonably concise. In what follows, Z is always a uniform commutative monoid, and \mathcal{E} is the smallest field of subsets of S containing $\mathcal{K} \cup \mathcal{G}$.

Definitions 3.1 $f, g \in \mathcal{F}$ are \mathcal{E} -separated if and only if there exist disjoint E, E' in \mathcal{E} such that f is supported by E and g is supported by E' ; ℓ is \mathcal{E} -additive if and only if $\ell(f + g) = \ell(f) + \ell(g)$, for all \mathcal{E} -separated f and g in \mathcal{F} . ℓ is **quasi-additive** if and only if, for each $W \in \text{unif } Z$ and $B \in \mathcal{B}$ there exists $V \in \text{unif } Z$ such that if $f, g \in \mathcal{F}_B$ and $(0, \ell(g)) \in V$ then $(\ell(f), \ell(f + g)) \in W$. The operator ℓ is **s-bounded over \mathcal{B}** if and only if, for each $B \in \mathcal{B}$, $W \in \text{unif } Z$ and disjoint sequence G in \mathcal{G} , there exists $m \in \mathbf{N}$ such that $(\ell(f), \ell(f + g)) \in W$, for all $n > m$ and $f, g \in \mathcal{F}_B$ with g supported by G_n . The operator ℓ is a **Riesz integral over \mathcal{R}** if and only if \mathcal{R} is a Riesz system, $(S, (\mathcal{K}, \mathcal{G}), (X, \mathcal{B}), (\mathcal{F}, \mathcal{T}))$, ℓ is \mathcal{E} -additive, s-bounded over \mathcal{B} , and, for each $B \in \mathcal{B}$, uniformly continuous on \mathcal{F}_B , and maps \mathcal{F}_B onto a relatively complete subset of Z .

Notation 3.2 $S(W, B, \ell) := \{\alpha \in \mathcal{G} : (\ell(f), \ell(f + g)) \in W, \text{ for all } f, g \in \mathcal{F}_B \text{ with } g \prec \alpha\}$.

We note that, if ℓ is quasi-additive, then s-boundedness of ℓ over \mathcal{B} is equivalent to the following property of ℓ : for all $B \in \mathcal{B}$, if g is a sequence in \mathcal{F}_B for which there exists a disjoint sequence G in \mathcal{G} such that g_n is supported by G_n for each n , then $\ell \circ g$ is a null sequence in Z . The following observations will be useful.

Lemma 3.3 Let ℓ be \mathcal{E} -additive. For each $B \in \mathcal{B}$, if g is a sequence in \mathcal{F}_B such that g_m and g_n are \mathcal{K} -separated for all $m, n \in \mathbf{N}$ with $m \neq n$, then the family of finite sums $\{\sum_{j \in J} \ell(g_j) : J \text{ is a finite subset of } \mathbf{N}\}$ is a subset of $\ell^\wedge \mathcal{F}_B$.

Proof. For such g and finite $J \subseteq \mathbf{N}$, $\sum_{i \in J} \ell(g_i) = \ell(\sum_{i \in J} g_i) \in \ell^\wedge \mathcal{F}_B$. \square

Lemma 3.4 If ℓ is an \mathcal{E} -additive, quasi-additive map, which, for all $B \in \mathcal{B}$, maps \mathcal{F}_B into a perfect subset of Z , then ℓ is s-bounded over \mathcal{B} .

Proof. Suppose that ℓ is not s -bounded over \mathcal{B} . Then there exists $B \in \mathcal{B}$, $W \in \text{unif}Z$ and a disjoint sequence $G \subseteq \mathcal{G}$, such that for all $m \in \mathbf{N}$ we can find $n > m$ such that $G_n \notin S(W, B, \ell)$. Since ℓ is quasi-additive, there exists $V \in \text{unif}Z$ such that for all $f, g \in \mathcal{F}_B$, if $(0, \ell(g)) \in V$ then $(\ell(f), \ell(f+g)) \in W$. We can find sequences f, g in \mathcal{F}_B such that g_j is supported by G_{n_j} , $n_i < n_j$ if $i < j$, and $(\ell(f_j), \ell(f_j + g_j)) \notin W$. Since $\ell^\wedge \mathcal{F}_B$ is perfect, we have that $\ell \circ g$ is a summable null sequence in Z [Lemma 3.3]. Hence, for all sufficiently large j , $(0, \ell(g_j)) \in V$, and therefore $(\ell(f_j), \ell(f_j + g_j)) \in W$, contradicting the choice of the sequences f and g . \square

Since a continuous linear map is necessarily \mathcal{E} -additive, quasi-additive, and uniformly continuous, then, as a consequence of Lemma 3.4, we have the following important result.

Let $\mathfrak{R} = (S, (\mathcal{K}, G), (X, \mathcal{B}), (\mathcal{F}, T))$ be a Riesz System in which Z and \mathcal{F} are topological vector spaces, with every bounded subset of Z being relatively complete and perfect. Every continuous linear map, ℓ , from \mathcal{F} to Z is an integral over \mathfrak{R} .

The following remarks indicate just how wide the family of integrals is.

Remarks 3.5

- .1 If B is any bounded subset of a topological vector space X , then \mathcal{F}_B is a bounded subset of (\mathcal{F}, T) .
- .2 If X, Z are topological vector spaces, and ℓ is continuous and linear, then it is quasi-additive, \mathcal{K} -additive, and maps bounded sets into bounded sets.
- .3 Every relatively weakly complete, bounded subset of a locally convex space is perfect. (Let Z be a locally convex space, and z be a sequence in Z for which the family of finite partial sums is bounded and relatively weakly complete. Then, for each $w \in Z'$, there exists M_w with $|\sum_{n \in J} \langle z_n, w \rangle| \leq M_w$, for all finite $J \subseteq \mathbf{N}$, and therefore $\sum_{n \in J} |\langle z_n, w \rangle| \leq 4M_w$, for all finite $J \subseteq \mathbf{N}$. Hence $\sum_{n \in \mathbf{N}} |\langle z_n, w \rangle| < \infty$, and consequently $(\sum_{n \in J} \langle z_n, w \rangle)_{J \subseteq \mathbf{N}, J \text{ finite}}$ is a weak Cauchy net in Z , and therefore converges to some point of Z . Thus z is weakly summable, and, by the Orlicz-Pettis theorem [?], p. 318, therefore summable.)
- .4 Every bounded subset of a semireflexive locally convex space is relatively weakly complete ([?], p.144).
- .5 Every quasi-complete nuclear locally convex space is semireflexive ([?], p.144).
- .6 Let $(S, \mathfrak{S}, \lambda)$ be a finite measure space. Denote by $L_0(\lambda)$ the space of all λ -equivalent classes of real-valued, \mathfrak{S} -measurable functions on S , with the topology of convergence in measure. By a theorem of Orlicz, every bounded subset of $L_0(\lambda)$ is perfect, [?], Theorem 5.1, [?], Theorem 8.
- .7 Let X, Z be topological vector spaces. With the notation of Theorem 3.3, let $\mathcal{F} \subseteq \mathcal{C}_p(S, X)$ be a module over $\mathcal{C}_p(S)$ which contains $X \otimes \mathcal{C}_K(S)$ (cf. Remark 2.5.1). If $\ell : \mathcal{C}_p(S, X) \rightarrow Z$ is linear and, for each $x \in X$, the partial operator ℓ_x on $\mathcal{C}_p(S)$ to Z given by $\ell_x(f) = \ell(xf)$ maps bounded sets into relatively compact subsets of Z , then ℓ maps \mathcal{F}_B into a relatively compact subset of Z , for all $B \in \mathcal{B}_0$. To see this, let $B \in \mathcal{B}_0$. There exists a finite dimensional subspace E of X such that $B \subseteq E$. Then \mathcal{F}_B is a bounded subset of $\mathcal{C}_p(S, E)$. Let $\{x_0, \dots, x_{n-1}\}$ be a linearly

independent basis for E , p_i the projection $\sum_{j < n} a_j x_j \rightarrow a_i$ on E to \mathbf{K} , and π_i the map on $\mathcal{C}_p(S, E)$ to $\mathcal{C}_p(S)$ given by $\pi_i(f) = p_i \circ f$, for each $i < n$. Then π_i is a continuous linear operator, and therefore maps bounded subsets of $\mathcal{C}_p(S, E)$ into bounded subsets of $\mathcal{C}_p(S, K)$. Then, $\ell^\wedge \mathcal{F}_B \subseteq \sum_{j < n} \ell_{x_j}^\wedge (\pi_j^\wedge \mathcal{F}_B)$, and is relatively compact, since a finite sum of relatively compact subsets of Z is again relatively compact [?], p. 26.

- .8 If A is a bounded subset of a locally convex space Z , then the bipolar $A^{00} \subseteq Z'' = ((Z')_\beta)'$ is $\sigma(Z'', Z')$ -compact. Thus the canonical embedding $\iota : Z \rightarrow Z''$ maps each bounded subset of Z into a relatively $\sigma(Z'', Z')$ -compact (therefore relatively $\sigma(Z'', Z')$ -complete) subset of Z'' [?, ?].

We note that the topological vector spaces of (1), (2), (6) and (7) above need not have a non-trivial continuous dual [?].

Examples 3.6 Suppose that \mathfrak{R} is a Riesz system, $(S, (\mathcal{K}, \mathcal{G}), (X, \mathcal{B}), (\mathcal{F}, \mathcal{T}))$, in which S is quasi-normal under $(\mathcal{K}, \mathcal{G})$, X and Z are topological vector spaces (with Z being a locally convex topological vector space in Examples 1–5), \mathcal{B} is the family of closed, balanced, totally bounded subsets of X , $\mathcal{F} \subseteq \mathcal{C}_p(S, X)$ is a topological vector space under \mathcal{T} , satisfying the conditions given in Remark 2.5.1.

- .1 Suppose that Z is a locally convex topological vector space. Let ℓ be a continuous linear operator on \mathcal{F}_0 to Z , which maps \mathcal{F}_B into a relatively complete subset of Z for each $B \in \mathcal{B}_0$. If the partial operators $\ell_x, x \in X$, map bounded sets into relatively weakly compact subsets, then ℓ is an integral over the Riesz system $\mathfrak{R}_0 = (S, (\mathcal{K}, \mathcal{G}), (X, \mathcal{B}_0), (\mathcal{F}_0, \mathcal{T}))$ [?].
- .2 Suppose that Z is a locally convex topological vector space. Let ℓ be a continuous linear operator on \mathcal{F} to Z . If ℓ maps bounded sets into relatively complete, relatively weakly compact sets, then ℓ is an integral over \mathfrak{R} [?].
- .3 Suppose that Z is a locally convex topological vector space. Let ℓ be a continuous linear operator on \mathcal{F} to Z . If ι is the natural embedding of Z into Z'' , then $\iota \circ \ell$ is an integral over \mathfrak{R} [?].

Suppose now that S is completely regular, with Stone-Ćech compactification βS [?, ?], and that Z is locally convex. Let $\mathcal{F} = \mathcal{C}_p(S, X)$ and $\mathcal{E} = \mathcal{C}(\beta S, X)$, with respectively the uniformities of uniform convergence on S and on βS , and let ℓ be a continuous linear operator on \mathcal{F} . Then the map $\pi_\beta : \mathcal{C}(\beta S, X) \rightarrow \mathcal{C}_p(S, X)$ is continuous and linear, and the composition of ℓ with π_β is a continuous linear operator, $\beta\ell$, which maps \mathcal{E}_B into a bounded subset of Z , and thus $\iota \circ \beta\ell$ maps \mathcal{E}_B into a relatively compact subset of $(Z'', \sigma(Z'', Z'))$, for each $B \in \mathcal{B}$. Since βS is compact Hausdorff and therefore locally compact, it has a Riesz system \mathfrak{R}^β , as given in Example 2.4.2. Hence, by Theorem 3.4 and Remark 3.5.3,

- .4 If ι is the embedding of Z into Z'' , then $\iota \circ \beta\ell$ is an integral over \mathfrak{R}^β , for every continuous, linear $\ell : \mathcal{C}_p(S, X) \rightarrow Z$.

The definition of integrals may be applied to dominated operators [?, ?]. Let X, Z be normed spaces, with norms $\|\cdot\|_X$ and $\|\cdot\|_Z$ respectively. Let S be a topological space. An operator $\ell : \mathcal{F} \rightarrow Z$ is **dominated** if there exists a positive measure, μ , σ -additive on the σ -algebra of Borel subsets of S , such that, for all $f \in \mathcal{F}$,

$$\|\ell(f)\|_Z \leq \int_S \|f(s)\|_X d\mu(s).$$

.5 If \mathcal{T} is the uniformity on \mathcal{F} of uniform convergence on S , Z is a Banach space, and ℓ is linear and dominated, then ℓ is an integral over \mathbb{R} .

Finally, we show that certain non-linear operators are integrals. Let \mathcal{E} be the smallest field containing $\mathcal{K} \cup \mathcal{G}$. We say that ℓ has the **Hammerstein property relative to \mathcal{E}** [?, ?] iff $\ell(f + g_1 + g_2) + \ell(f) = \ell(f + g_1) + \ell(f + g_2)$, for all f, g_1, g_2 in \mathcal{F} with g_1 and g_2 being \mathcal{E} -separated. (If it is linear, then ℓ trivially has the Hammerstein property relative to \mathcal{E} .)

.6 Suppose that X, Z, \mathcal{F} are topological vector spaces. If ℓ has the Hammerstein property relative to some \mathcal{E} , and, for each $B \in \mathcal{B}$, ℓ is uniformly continuous on \mathcal{F}_B , and maps it into a quasi-perfect, relatively complete subset of Z , then ℓ is an integral over \mathbb{R} .

Noting that ℓ is necessarily \mathcal{E} -additive if it has the Hammerstein property over \mathcal{E} , the proof is based on the following lemmas [?]. (We suppose that \mathcal{F} satisfies the conditions of Remark 2.5.1.)

Lemma 3.7 Suppose that X is a topological vector space. If $\kappa \in \mathcal{K}$, $\gamma \in \mathcal{G}$, $g \in \mathcal{F}$ with $\kappa \subseteq \gamma$, $g \prec \kappa$, $p \in \mathcal{C}(S, [0, 1])$, with p identically 1 on κ and 0 on $S \setminus \gamma$, then, for all $f \in \mathcal{F}$,

$$\ell(pf + g) + \ell(f) = \ell(pf) + \ell(f + g).$$

Proof. Since g and $(p - 1)f$ are \mathcal{E} -separated, then,

$$\begin{aligned} \ell(pf + g) + \ell(f) &= \ell(f + (p - 1)f + g) + \ell(f) \\ &= \ell(f + (p - 1)f) + \ell(f + g) \\ &= \ell(pf) + \ell(f + g). \end{aligned}$$

□

Lemma 3.8 If X, Z, \mathcal{F} are topological vector spaces, then ℓ is s -bounded over \mathcal{B} .

Proof. If not, then there exist $B \in \mathcal{B}$, $W \in \text{unif} Z$, disjoint sequences K in \mathcal{K} and G in \mathcal{G} , and sequences f, g in \mathcal{F}_B , such that, for each $n \in \mathbb{N}$, $K_n \subseteq G_n$, $g_n \prec K_n$ and $(\ell(f_n + g_n), \ell(f_n)) \notin W$. Choose $C \in \mathcal{B}$ such that $B + B \subseteq C$, and $V \in \text{unif} Z$ such that $V \circ V \subseteq W$. By the properties of $(\mathcal{K}, \mathcal{G})$, we can choose sequences K' in \mathcal{K} and G' in \mathcal{G} such that $K_n \subseteq G'_n \subseteq K'_n \subseteq G_n$, and functions p_n on S to $[0, 1]$ such that p_n equals 1 on G'_n and 0 on $S \setminus K'_n$. Then $(\ell(p_n f_n + g_n), \ell(p_n f_n)) \notin W$, for all $n \in \mathbb{N}$. However the functions $p_m f_m$ and $p_n f_n$ are separated by \mathcal{E} , for $m \neq n$, and likewise $p_m f_m + g_m$, $p_n f_n + g_n$. Moreover, these functions are all in \mathcal{F}_C . Thus, the sets

$$\left\{ \sum_{n \in J} \ell(p_n f_n + g_n) : J \subseteq \mathbb{N} \text{ is finite} \right\} = \left\{ \ell \left(\sum_{n \in J} (p_n f_n + g_n) \right) : J \subseteq \mathbb{N} \text{ is finite} \right\},$$

$$\left\{ \sum_{n \in J} \ell(p_n f_n) : J \subseteq \mathbf{N} \text{ is finite} \right\} = \left\{ \ell \sum_{n \in J} (p_n f_n) : J \subseteq \mathbf{N} \text{ is finite} \right\}$$

are quasi-perfect. Hence the sequences $(\ell(p_n f_n + g_n))_{n \in \mathbf{N}}$, $(\ell(p_n f_n))_{n \in \mathbf{N}}$ are quasi-summable, contradicting the choice of f_n , g_n and p_n above. \square

The foregoing lead to the following important assertion.

Lemma 3.9 *Let X, Z, \mathcal{F} be topological vector spaces. Suppose that ℓ is quasi-additive and, for each $B \in \mathcal{B}$, is uniformly continuous on \mathcal{F}_B , and sends \mathcal{F}_B into a quasi-perfect, relatively complete subset of Z . If ℓ has the Hammerstein property relative to \mathcal{E} then ℓ is an integral over \mathfrak{R} .*

Proof. By the definition of integral (Definition 3.1), and Lemma 3.4. \square

As observed earlier in Remark 3.5.9, when Z is a locally convex topological vector space we have a continuous embedding $\iota : Z \rightarrow Z''$, which carries each bounded subset of Z into a relatively $\sigma(Z'', Z')$ -compact (therefore $\sigma(Z'', Z')$ -complete) subset of Z'' . Thus, for locally convex Z , we conclude that $\iota \circ \ell$ is an integral over \mathfrak{R} , whenever ℓ has the Hammerstein property, and for each $B \in \mathcal{B}$, ℓ is uniformly continuous on \mathcal{F}_B , and maps it into a bounded subset of Z . More generally, let S be completely regular, \mathcal{F} be as in Example 3.6.6 above, and ℓ be an operator on \mathcal{F} to Z with the Hammerstein property, uniformly continuous on \mathcal{F}_B , and mapping it into a bounded subset of Z , for all $B \in \mathcal{B}$. Then $\iota \circ \beta \ell$ is an integral over \mathfrak{R}^β , where βS is the Stone-Ćech compactification of S , π_β is the restriction map $\mathcal{C}(\beta S, X) \rightarrow \mathcal{C}_p(S, X)$, $\beta \ell = \ell \circ \pi_\beta$, and \mathfrak{R}^β is the Riesz system determined by βS (Example 3.6.4).

It will be shown that all integrals over Riesz systems do in fact have integral representations.

4 Riesz Measures

Throughout this section, X and Z are uniform commutative monoids, \mathfrak{R} is a Riesz system, $(S, (\mathcal{K}, \mathcal{G}), (X, \mathcal{B}), (\mathcal{F}, \mathcal{T}))$, “integral” stands for “ Z -valued Riesz integral over \mathfrak{R} ”, and ℓ is an integral. It will be seen that each integral ℓ generates an additive, Z^X -valued function on \mathcal{K} . This function extends to a $(\mathcal{K}, \mathcal{G})$ -regular set function $[\cdot, \cdot, \cdot]$, ν_ℓ , defined for all subsets of S , which is additive on a field containing $\mathcal{K} \cup \mathcal{G}$. In this section we address the construction of ν_ℓ and the determination of its characteristic properties. These properties lead naturally to the concept of a **Riesz measure** [Definition 4.11]. Informally, integrals generate Riesz measures, Riesz measures generate integrals, and the correspondence is one-to-one [Theorem 5.1].

Notation 4.1 *For all $\kappa \in \mathcal{K}, \gamma \in \mathcal{G}, f \in \mathcal{F}, B \subseteq X, x \in X$, and $W \in \text{unif } Z$:*

$$\begin{aligned} \mathcal{K}^-(\gamma) &:= \{\alpha \in \mathcal{K} : \alpha \subseteq \gamma\}, \text{ directed by } \subseteq, \\ \mathcal{G}^+(\kappa) &:= \{\beta \in \mathcal{G} : \beta \supseteq \kappa\}, \text{ directed by } \supseteq, \\ \mathcal{F}(\kappa, \gamma, B) &:= \{f \in \mathcal{F}_B : f \prec \gamma \setminus \kappa\}, \\ \mathcal{F}(x, \kappa, \gamma, B) &:= \{f \in \mathcal{F}_B : \kappa =_x f \text{ and } f \prec \gamma\}, \\ Z_{\ell, B} &:= (\ell^\wedge \mathcal{F}_B)^{cl} \\ Z_{\ell, x} &:= \{\ell(f) : f \in \mathcal{F}, \text{ rng } f \subseteq \mathcal{B}\text{-hull}(x)\}^{cl} \end{aligned}$$

Propositions 4.2 For all $\kappa \in \mathcal{K}, \gamma \in \mathcal{G}, B \in \mathcal{B}$, and $x \in B$:

- .1 For all $W \in \text{unif } Z$, there exists $\kappa' \in \mathcal{K}^-(\gamma)$ such that $\gamma \setminus \kappa' \in S(W, B, \ell)$.
Hence $\{\ell^\wedge \mathcal{F}_*(\alpha, \gamma, B) : \alpha \in \mathcal{K}^-(\gamma)\}$ is a filter base in Z converging to 0.
- .2 For all $W \in \text{unif } Z$, there exists $\gamma' \in \mathcal{G}^+(\kappa)$ such that $\gamma' \setminus \kappa \in S(W, B, \ell)$.
Hence $\{\ell^\wedge \mathcal{F}_*(\kappa, \beta, B) : \beta \in \mathcal{G}^+(\kappa)\}$ is a filter base in Z converging to 0.
- .3 $\{\ell^\wedge \mathcal{F}^*(x, \kappa, \beta, B) : \beta \in \mathcal{G}^+(\kappa)\}$ is a Cauchy filter base in $Z_{\ell, B}$.

Note that the filter bases are non-empty (by Assumptions (3), (4) and (11)), and that $Z_{\ell, B}$ is a complete subset of Z for each $B \in \mathcal{B}$.

Proofs.

- .1 If not, there exist $V \in \text{unif } Z$, and for each $\alpha \in \mathcal{K}^-(\gamma)$ an $f \in \mathcal{F}_B$, and $g \in \mathcal{F}_*(\alpha, \gamma, B)$, such that $(\ell(f), \ell(f+g)) \notin V$. Recursively define sequences φ in \mathcal{F}_B , η and η' in \mathcal{K} , γ' in \mathcal{G} , χ such that $\chi_n \in \mathcal{F}_*(\eta_n, \gamma, B)$, such that for all $n \in \mathbb{N}$: $\eta_0 \subseteq \gamma'_0 \subseteq \eta'_0 \subseteq \gamma$, $\eta_{n+1} \subseteq \gamma'_{n+1} \subseteq \eta'_{n+1} \subseteq \gamma \setminus \bigcup_{i \leq n} \eta'_i$, χ_n is supported by η_n , and $(\ell(\varphi_n), \ell(\varphi_n + \chi_n)) \notin V$. Since ℓ is s -bounded over \mathcal{B} , this yields a contradiction. \square
- .2 Similarly. \square
- .3 Let $W \in \text{unif } Z$. There exist $C \in \mathcal{B}$, $V \in \text{unif } Z$, $T \in \mathcal{T}, U \in \text{unif } X$, and $\gamma \in \mathcal{G}^+(\kappa)$, such that: $B+B \subseteq C$, $V \circ V \circ V \subseteq W$; $\gamma \setminus \kappa \in S(V, B, \ell)$ (by Proposition 4.2.2 above); if $f, g \in \mathcal{F}_C$ and $(f, g) \in T$ then $(\ell(f), \ell(g)) \in V$ (by uniform continuity of ℓ on \mathcal{F}_C); and $T \text{ ext}_B U$ (by Assumption (12)). Let $f_1, f_2 \in \mathcal{F}^*(x, \kappa, \gamma, B)$. There exist $\beta_1, \beta_2 \in \mathcal{G}^+(\kappa)$ such that $\beta_1 \cup \beta_2 \subseteq \gamma$, and $f_i(s) = x$ for all $s \in \beta_i$, $i = 1, 2$. Hence $(f_1(s), f_2(s)) \in U$ for all $s \in \beta_1 \cap \beta_2$, and $\kappa \subseteq \beta_1 \cap \beta_2 \subseteq \gamma$. Thus, by Assumption (12), there exist $p_1, p_2 \in \mathcal{F}_*(\kappa, \gamma, B)$ such that $(f_1 + p_1, f_2 + p_2) \in T$, and p_i is supported by $\gamma \setminus \kappa$, $i = 1, 2$. Then, for $i = 1, 2$, $f_i + p_i \in \mathcal{F}_C$, and therefore

$$(\ell(f_1), \ell(f_2)) = (\ell(f_1), \ell(f_1 + p_1)) \circ (\ell(f_1 + p_1), \ell(f_2 + p_2)) \circ (\ell(f_2 + p_2), \ell(f_2)) \in V \circ V \circ V \subseteq W. \quad \square$$

Proposition 4.3 If $A, B \in \mathcal{B}$ are such that $A + A \subseteq B$, and, for all $i = 1, 2$, $V_i \in \text{unif } Z$, and $\gamma_i \in S(V_i, B, \ell)$, then $\gamma_1 \cup \gamma_2 \in S((V_1 \circ V_2)^{cl}, A, \ell)$.

Proof. Given any $W \in \text{unif } Z$, choose $W_1 \in \text{unif } Z$ such that $W_1^3 \subseteq W$. By uniform continuity of ℓ on \mathcal{F}_B , there exists $T \in \text{unif } \mathcal{F}$ such that for all $f, g \in \mathcal{F}_B$, if $(f, g) \in T$ then $(\ell(f), \ell(g)) \in W_1$. By Assumption (12), choose $U \in \text{unif } X$ such that $T \text{ ext}_B U$.

Let $\gamma_i \in \mathcal{G} \cap S(V_i, B, \ell)$ for $i = 1, 2$ and $f, h \in \mathcal{F}_A$, with $h \prec \gamma_1 \cup \gamma_2$. There exist $\kappa_1 \in \mathcal{K}$ such that $\kappa_1 \subseteq \gamma_1 \cup \gamma_2$ and h equals 0 on $S \setminus \kappa_1$. Now, by Proposition 4.2.1 above, there exists $\alpha \in \mathcal{K}$ with $\kappa_1 \subseteq \alpha \subseteq \gamma_1 \cup \gamma_2$ such that $(\gamma_1 \cup \gamma_2) \setminus \alpha \in S(W_1, B, \ell)$. By Assumptions (1)–(4), there exist $\kappa' \in \mathcal{K}$ and $\gamma' \in \mathcal{G}$ such that $\alpha \subseteq \gamma' \subseteq \kappa' \subseteq \gamma_1 \cup \gamma_2$. By Assumption (13), there exist $g_1, g_2 \in \mathcal{F}_A$ such that g_i is supported by γ_i for $i = 1, 2$, $g_1 + g_2 \in \mathcal{F}_A$, and $h(s) = g_1(s) + g_2(s)$ for all $s \in \kappa'$. Thus $(h(s), g_1(s) + g_2(s)) \in U$ for all $s \in \gamma'$. By Assumption (12), there exist $p, q \in \mathcal{F}_A$, both supported by $(\gamma_1 \cup \gamma_2) \setminus \alpha$, such that $(g_1 + g_2 + q, h + p) \in T$. Hence, by the above choices,

$$\begin{aligned}
& (\ell(f), \ell(f+h)) \\
&= (\ell(f), \ell(f+g_1)) \circ (\ell(f+g_1), \ell(f+g_1+g_2)) \circ \\
&\quad (\ell(f+g_1+g_2), \ell(f+g_1+g_2+q)) \circ \\
&\quad (\ell(f+g_1+g_2+q), \ell(f+h+p)) \circ (\ell(f+h+p), \ell(f+h)) \\
&\varepsilon V_1 \circ V_2 \circ W_1 \circ W_1 \circ W_1 \subseteq V_1 \circ V_2 \circ W.
\end{aligned}$$

The result follows by Theorem 7, p.179, of [?]. \square

Corollary 4.4 *Let $W \in \text{unif } Z$, $A, B \in \mathcal{B}$, and $A + A \subseteq B$. If $V \in \text{unif } Z$ is such that $V^3 \subseteq W$, and $\gamma_1, \gamma_2 \in S(V, B, \ell)$ for $i = 1, 2$, then $\gamma_1 \cup \gamma_2 \in S(W, A, \ell)$.*

We note that, for all $\kappa \in \mathcal{K}$, $x \in X$ and $B \in \mathcal{B}$ with $x \in B$, the Cauchy filter base, $\{\ell^\wedge \mathcal{F}^*(x, \kappa, \beta, B) : \beta \in \mathcal{G}^+(\kappa)\}$, is convergent to some point $\tau_\ell(\kappa, x)$ in $Z_{\ell, x} \subseteq Z_{\ell, \mathcal{B}}$, since, by the definition of integral, this is a complete subset of Z . Further, by Proposition 4.8.6 below, for all $\gamma \in \mathcal{G}$ and $x \in X$, the net $(\tau_\ell(\kappa, x) : \kappa \in \mathcal{K}^-(\gamma))$ is Cauchy in $Z_{\ell, \mathcal{B}}$, and therefore converges to some point $\xi_\ell(\gamma, x)$ in $Z_{\ell, \mathcal{B}}$.

Definition 4.5

$$(\tau_\ell(\kappa))(x) = \tau_\ell(\kappa, x), \quad \xi_\ell(\gamma) = \lim_{\kappa \in \mathcal{K}, \kappa \subseteq \gamma} \tau_\ell(\kappa), \quad \text{for all } \gamma \in \mathcal{G},$$

$$\nu_\ell(\alpha) = \lim_{\gamma \in \mathcal{G}, \alpha \subseteq \gamma} \xi_\ell(\gamma), \quad \text{for all } \alpha \subseteq S \text{ for which the limit exists.}$$

Clearly, the Cauchy filter base, $\{\ell^\wedge \mathcal{F}^*(x, \kappa, \beta, \mathcal{B}\text{-hull}(x)) : \beta \in \mathcal{G}^+(\kappa)\}$, converges to $(\tau_\ell(\kappa))(x)$. We shall see below that ν_ℓ is defined in fact for all subsets of S .

Definition 4.6 *For any function h on \mathcal{K} to Z^X , $B \in \mathcal{B}$ and $W \in \text{unif } Z$, a subset α of S is W -small with respect to (h, B) if and only if there exists $\gamma \in \mathcal{G}$, with $\alpha \subseteq \gamma$, such that $(\sum_{\kappa \in K} x_\kappa \cdot h(\kappa), \sum_{\kappa \in K} (x_\kappa + y_\kappa) \cdot h(\kappa)) \in W$, for all finite, disjoint $K \subseteq \mathcal{K}$ with $\bigcup K \subseteq \gamma$, and all functions x, y on K to B .*

Proposition 4.7.3 below characterizes W -smallness in terms of the function family \mathcal{F} and the operator ℓ .

Propositions 4.7 *Let $A \in \mathcal{B}, \gamma \in \mathcal{G}, W \in \text{unif } Z$.*

- .1 *If K is a finite, disjoint subfamily of \mathcal{K} with $\bigcup K \subseteq \gamma$, and x is a function on K to B , then there exists $h \in \mathcal{F}_B$, supported by γ , with $h(t) = x_\kappa$, for all $\kappa \in K$ and $t \in \kappa$, such that*

$$(\ell(h), \sum_{\kappa \in K} x_\kappa \cdot \tau_\ell(\kappa)) \in W.$$

- .2 *For each $h \in \mathcal{F}_B$ supported by γ there exist a finite, disjoint $K \subseteq \mathcal{K}$ with $\bigcup K \subseteq \gamma$, such that for each choice function s on K ,*

$$(\ell(h), \sum_{\kappa \in K} h(s_\kappa) \cdot \tau_\ell(\kappa)) \in W.$$

- .3 *For each $B \in \mathcal{B}$, γ is W -small with respect to (τ_ℓ, B) if and only if $\gamma \in S(W, B, \ell)$*

Proofs. 4.7

- .1 By the separation properties of \mathcal{K} and \mathcal{G} , and the definition of τ_ℓ , there exist functions G on K to \mathcal{G} , g on K to \mathcal{F} , such that G has disjoint range, $\kappa \subseteq G_\kappa \subseteq \gamma$, $g_\kappa \in \mathcal{F}^*(x_\kappa, \kappa, G_\kappa, B)$ for all $\kappa \in K$, and

$$(\sum_{\kappa \in K} \ell(g_\kappa), \sum_{\kappa \in K} x_{\kappa} \cdot \tau_\ell(\kappa)) \in W.$$

By the \mathcal{K} -additivity of ℓ , $\sum_{\kappa \in K} \ell(g_\kappa) = \ell(\sum_{\kappa \in K} g_\kappa)$. Let h be the function $\sum_{\kappa \in K} g_\kappa$. Then $\text{rng } h \subseteq B$. \square

- .2 Choose $V \in \text{unif } Z$ such that $V^4 \subseteq W$, and $C \in \mathcal{B}$ such that $B + B \subseteq C$. Choose $T \in \mathcal{T}$ such that for all $f, g \in \mathcal{F}_C$, if $(f, g) \in T$ then $(\ell(f), \ell(g)) \in V$, and choose $U \in \text{unif } X$ such that $\text{Text}_B U$ [Assumption (12)]. Let h be any function in \mathcal{F}_B which is supported by γ . Since h is finitely \mathcal{G} -partitionable [Assumption (10)], there exists a finite $G \subseteq \mathcal{G}$ with $S = \bigcup G$, such that for all $\alpha \in G$ and $x, y \in \alpha$, $(h(x), h(y)) \in U$. Let $H = \{\alpha \cap \gamma : \alpha \in G\}$, and list H as $\{H_0, \dots, H_{n-1}\}$ for some $n \in \mathbb{N}$. Clearly, $\bigcup H = \gamma$. Choose a finite sequence P in $\text{unif } Z$, and, by Corollary 4.4, a finite sequence B' in \mathcal{B} , such that $P_0 \circ P_0 \subseteq V$, and if $\gamma_i \in S(P_0, B'_0, \ell)$ for $i = 1, 2$, then $\gamma_1 \cup \gamma_2 \in S(V, A, \ell)$, $P_{j+1} \circ P_{j+1} \subseteq P_j$, and if $\gamma_i \in S(P_{j+1}, B'_{j+1}, \ell)$ for $i = 1, 2$, then $\gamma_1 \cup \gamma_2 \in S(P_j, B_j, \ell)$, for $0 \leq j < n-1$. By Proposition 4.2.1 and Assumption (4), construct finite sequences κ, η in \mathcal{K} , and β, δ in \mathcal{G} , inductively as follows:

$$\begin{aligned} \delta_0 &= H_0, \delta_0 \setminus \kappa_0 \in S(P_0, B'_0, \ell), \text{ and } \kappa_0 \subseteq \beta_0 \subseteq \eta_0 \subseteq \delta_0. \\ \delta_{i+1} &= H_{i+1} \setminus \bigcup_{j \leq i} \eta_j, \delta_{i+1} \setminus \kappa_{i+1} \in S(P_{i+1}, B'_{i+1}, \ell), \text{ and} \\ \kappa_{i+1} &\subseteq \beta_{i+1} \subseteq \eta_{i+1} \subseteq \delta_{i+1}, \text{ for } i < n-1. \end{aligned}$$

For each $i < n$, and $s_i \in \kappa_i$, choose $f_i \in \mathcal{F}^*(h(s_i), \kappa_i, \beta_i, A)$ such that

$$(\sum_{i < n} \ell(f_i), \sum_{i < n} h(s_i) \cdot \tau_\ell(\kappa_i)) \in W.$$

Let $g = \sum_{i < n} f_i$. Then $\ell(g) = \sum_{i < n} \ell(f_i)$, $\text{rng } g \subseteq B$, g is supported by γ and, for some $\omega \in \mathcal{G}$ with $\bigcup_{i < n} \kappa_i \subseteq \omega$, we have that $(h(t), g(t)) \in U$ for all $t \in \omega$. Thus, by Assumption (12), there exists $p, q \in \mathcal{F}_B$ supported by $\gamma \setminus \bigcup \kappa$ such that $(h + p, g + q) \in T$. Now, $\gamma \setminus \bigcup \kappa \subseteq \bigcup_{i < n} (\delta_i \setminus \kappa_i)$, and therefore, by repeated application of Corollary 4.4, $\gamma \setminus \bigcup \kappa \in S(V, B, \ell)$. Thus,

$$\begin{aligned} (\ell(h), \sum_{i \leq 1} h(s_i) \cdot \tau_\ell(\kappa_i)) &= (\ell(h), \ell(h + p)) \circ (\ell(h + p), \ell(g + q)) \circ \\ &(\ell(g + q), \ell(g)) \circ (\ell(g), \sum_{i < n} h(s_i) \cdot \tau_\ell(\kappa_i)) \in V^4 \subseteq W \end{aligned} \quad \square$$

- .3 Let $B \in \mathcal{B}$. We shall first show that if $\gamma \in S(W, B, \ell)$ then γ is W -small with respect to (τ_ℓ, B) . Let K be a finite, disjoint subfamily of \mathcal{K} with $\bigcup K \subseteq \gamma$. Let $V \in \text{unif } Z$, and x, y be functions on K to B . Choose γ' on K to \mathcal{G} , and for each $\kappa \in K$, let $f_\kappa \in \mathcal{F}(x_\kappa, \kappa, \gamma'_\kappa, B)$, $g_\kappa \in \mathcal{F}(\kappa, \gamma'_\kappa, B)$, such that $\text{rng } \gamma'$ is a disjoint subfamily of \mathcal{G} , with $\kappa \subseteq \gamma'_\kappa \subseteq \gamma$ for each $\kappa \in K$, and

$$(\sum_{\kappa \in K} x_{\kappa} \cdot \tau_\ell(\kappa), \sum_{\kappa \in K} \ell(f_\kappa)) \in V,$$

$$(\sum_{\kappa \in K} \ell(f_\kappa), \sum_{\kappa \in K} \ell(f_\kappa + g_\kappa)) \in V$$

$$(\sum_{\kappa \in K} (x_\kappa + y_\kappa) \cdot \tau_\ell(\kappa), \sum_{\kappa \in K} \ell(f_\kappa + g_\kappa)) \in V.$$

Then, by the above choices and the \mathcal{E} -additivity of ℓ ,

$$\begin{aligned} & (\sum_{\kappa \in K} x_\kappa \cdot \tau_\ell(\kappa), \sum_{\kappa \in K} (x_\kappa + y_\kappa) \cdot \tau_\ell(\kappa)) \\ &= (\sum_{\kappa \in K} x_\kappa \cdot \tau_\ell(\kappa), \sum_{\kappa \in K} \ell(f_\kappa)) \circ \\ & \quad (\ell(\sum_{\kappa \in K} f_\kappa), \ell(\sum_{\kappa \in K} (f_\kappa + g_\kappa))) \circ \\ & \quad (\sum_{\kappa \in K} \ell(f_\kappa + g_\kappa), \sum_{\kappa \in K} (x_\kappa + y_\kappa) \cdot \tau_\ell(\kappa)) \\ & \in V \circ W \circ V. \end{aligned}$$

Since $W = \bigcap_{V \in \text{unif} Z} V \circ W \circ V$ ([?], Theorem 7, p.179), it follows that if $\gamma \in S(W, B, \ell)$, then γ is W -small with respect to (τ_ℓ, B) . We shall now show the converse.

Suppose γ is W -small with respect to (τ_ℓ, B) . Let $f, g \in \mathcal{F}_B$ with $g \prec \delta \in \mathcal{K}$ and $\delta \subseteq \gamma$. Let $B_1, B_2 \in \mathcal{B}$ with $B + B \subseteq B_1$, $B_1 + B_1 \subseteq B_2$. There exists $V \in \text{unif} Z$ such that $V^9 \subseteq W$, $T \in \mathcal{T}$ such that $(\ell(h), \ell(h')) \in V$ for all $h, h' \in \mathcal{F}_{B_2}$ with $(h, h') \in T$, and $U \in \text{unif} X$ such that $T \text{ext}_{B_2} U$. Since $(X, +)$ is a uniform monoid, there exists $U' \in \text{unif} X$ with $U' \subseteq U$ such that if $(s_1, t_1) \in U'$ and $(s_2, t_2) \in U'$, then $(s_1 + s_2, t_1 + t_2) \in U$. By the finite \mathcal{G} -partitionability of f and g (Assumption (10)) choose a finite $G \subseteq \mathcal{G}$ such that $(f(s), f(s')) \in U'$ and $(g(s), g(s')) \in U'$ for all $s, s' \in \gamma' \in G$. Let $G = \{G_1, \dots, G_n\}$. By Corollary 4.4, there exist $V_1, V_2 \in \text{unif} Z$ with $V_2 \subseteq V_1 \subseteq W$, such that if $\gamma_1, \gamma_2 \in S(V_1, B_1, \ell)$ for $i = 1, 2$, then $\gamma_1 \cup \gamma_2 \in S(V, B, \ell)$, and if $\gamma_1, \gamma_2 \in S(V_2, B_2, \ell)$ for $i = 1, 2$, then $\gamma_1 \cup \gamma_2 \in S(V_1, B_1, \ell)$.

Using Corollary 4.4 and Proposition 4.2.1, choose finite, disjoint $K \subseteq \mathcal{K}$ such that $K_i \subseteq G_i \setminus \bigcup_{j < i} K_j$ and $\bigcup G \setminus \bigcup K \in S(V_2, B_2, \ell)$. Further choose $\kappa \in \mathcal{K}$ such that $\delta \subseteq \kappa \subseteq \gamma$ and $\gamma \setminus \kappa \in S(V_2, B_2, \ell)$. Let $K'_i = K_i \setminus \gamma$, $K''_i = K_i \cap \kappa$, $i = 1, \dots, n$, list the non-empty members of $\{K'_i\}_{i=1}^n \cup \{K''_i\}_{i=1}^n$ as $\{\eta_j\}_{j=1}^m$, and let $M := \{j : \eta_j \cap \gamma = \emptyset\}$, $N := \{j : \eta_j \subseteq \gamma\}$. For each $j \in M \cup N$ let $s_j \in \eta_j$, $x_j = f(s_j)$, $y_j = g(s_j)$ if $j \in N$ and $y_j = 0$ if $j \in M$.

Then there exists a finite disjoint $\{E_1, \dots, E_m\} \subseteq \mathcal{G}$, finite sequences $(h_j^1)_{j=1}^m$ in \mathcal{F}_B and $(h_j^2)_{j=1}^m$ in \mathcal{F}_B , such that

- (1) for each j there exists i_j such that $E_j \subseteq G_{i_j}$,
- (2) $\eta_j \subseteq E_j \subseteq S \setminus \kappa$ if $j \in M$, and $\eta_j \subseteq E_j \subseteq \gamma$ if $j \in N$,
- (3) $h_j^1 \in \mathcal{F}^*(x_j, \eta_j, E_j, B)$, $h_j^2 \in \mathcal{F}^*(x_j + y_j, \eta_j, E_j, B)$ such that, if $h_1 = \sum h_j^1$ and $h_2 = \sum h_j^2$, then

$$(\ell(h_1), \sum_{j \leq n} \tau_\ell(\eta_j, x_j)) \in V, \quad (\ell(h_2), \sum_{j \leq n} \tau_\ell(\eta_j, x_j + y_j)) \in V.$$

By Assumption (12), there exist $p_i, q_i \in \mathcal{F}_A$ with $p_i \prec (\bigcup G \setminus \bigcup K) \cup (\gamma \setminus \kappa)$ such that $(f + p_1, h_1 + q_1) \in T$ and $(f + g + p_2, h_2 + q_2) \in T$. Since $(\bigcup G \setminus \bigcup K) \cup (\gamma \setminus \kappa) \in S(V_1, B_1, \ell)$, then

$$\begin{aligned}
& (\ell(f), \ell(f + g)) \\
&= (\ell(f), \ell(f + p_1)) \circ (\ell(f + p_1), \ell(h_1 + q_1)) \circ (\ell(h_1 + q_1), \ell(h_1)) \circ \\
& \quad (\ell(h_1), \sum_{j=1}^m x_j \cdot \tau_\ell(\eta_j)) \circ (\sum_{j=1}^m x_j \cdot \tau_\ell(\eta_j), \sum_{j=1}^m (x_j + y_j) \cdot \tau_\ell(\eta_j)) \circ \\
& \quad (\sum_{j=1}^m (x_j + y_j) \cdot \tau_\ell(\eta_j), \ell(h_2)) \circ (\ell(h_2), \ell(h_2 + p_2)) \circ \\
& \quad (\ell(h_2 + p_2), \ell(f + g + q_2)) \circ (\ell(f + g + q_2), \ell(f + g)) \\
& \in V^9 \subseteq W. \quad \square
\end{aligned}$$

Hereafter we shall use the phrases “ $\gamma \in S(W, B, \ell)$ ” and “ γ is W -small with respect to (τ_ℓ, B) ” interchangeably. Essential properties of τ_ℓ are given below.

Propositions 4.8 *For all $\kappa \in \mathcal{K}$, $\gamma \in \mathcal{G}$, $B \in \mathcal{B}$ and $W \in \text{unif } Z$:*

- .1 *There exists $V \in \text{unif } Z$ such that if γ is V -small with respect to (τ_ℓ, B) , then $(x \cdot \tau_\ell(\kappa), x \cdot \tau_\ell(\kappa \setminus \gamma)) \in W$ for all $x \in B$.*
- .2 *There exists $U \in \text{unif } X$ such that for all finite disjoint $K \subseteq \mathcal{K}$ and functions x, y on K to B , if $(x_\kappa, y_\kappa) \in U$ for all $\kappa \in K$ then $(\sum_{\kappa \in K} x_\kappa \cdot \tau_\ell(\kappa), \sum_{\kappa \in K} y_\kappa \cdot \tau_\ell(\kappa)) \in W$.*
- .3 *For each disjoint sequence α in \mathcal{K} there exists $m \in \mathbb{N}$ such that, for all $n > m$, α_n is W -small with respect to (τ_ℓ, B) .*
- .4 *If $A, B \in \mathcal{B}$ are such that $A + A \subseteq B$, $V_1, V_2 \in \text{unif } Z$, and γ_i is V_i -small with respect to (τ_ℓ, B) , for $i = 1, 2$, then $\gamma_1 \cup \gamma_2$ is $V_1 \circ V_2 \circ W$ -small with respect to (τ_ℓ, B) .*
- .5 *There exists $\gamma' \in \mathcal{G}$, with $\kappa \subseteq \gamma'$, such that $(x \cdot \tau_\ell(\kappa), x \cdot \tau_\ell(\kappa')) \in W$, for all $\kappa' \in \mathcal{K}$ with $\kappa \subseteq \kappa' \subseteq \gamma'$, and all $x \in B$.*
- .6 *There exists $\eta \in \mathcal{K}$, with $\eta \subseteq \gamma$, such that $(x \cdot \tau_\ell(\eta), x \cdot \tau_\ell(\kappa')) \in W$, for all $\kappa' \in \mathcal{K}$ with $\eta \subseteq \kappa' \subseteq \gamma$, and all $x \in B$.*
- .7 *τ_ℓ is additive.*

Proofs. 4.8

- .1 Choose $V_0 \in \text{unif } Z$ such that $V_0^5 \subseteq W$; $C \in \mathcal{B}$ such that $B + B \subseteq C$; and $T \in \mathcal{T}$ such that, for all $f, g \in \mathcal{F}_C$, if $(f, g) \in T$ then $(\ell(f), \ell(g)) \in V$. By Assumption (12), there exists $U \in \text{unif } X$ such that $T \text{ ext}_B U$. Let $V_1 \in \text{unif } Z$ with $V_1^3 \subseteq V_0$. Suppose now that γ is V_0 -small with respect to (τ_ℓ, B) . Let $x \in B$. There exist $\gamma_1, \gamma_2 \in \mathcal{G}$ with $\gamma_2 \subseteq \gamma_1$, such that

$$\begin{aligned}
& \kappa \subseteq \gamma_1, \gamma_1 \setminus \kappa \text{ is } V_1\text{-small with respect to } (\tau_\ell, C), \text{ and } (\tau_\ell(x, \kappa), \ell(f)) \in V_0 \\
& \quad \text{for all } f \in \mathcal{F}^*(x, \kappa, \gamma_1, B), \\
& \kappa \setminus \gamma \subseteq \gamma_2, \gamma_2 \setminus (\kappa \setminus \gamma) \text{ is } V_1\text{-small with respect to } (\tau_\ell, C), \text{ and} \\
& \quad (\tau_\ell(x, \kappa \setminus \gamma), \ell(g)) \in V_0 \text{ for all } g \in \mathcal{F}^*(x, \kappa \setminus \gamma, \gamma_2, B).
\end{aligned}$$

Let $f_1 \in \mathcal{F}^*(x, \kappa, \gamma_1, B)$, and $f_2 \in \mathcal{F}^*(x, \kappa \setminus \gamma, \gamma_2, B)$. Then there exists $\beta \in \mathcal{G}$ such that $\kappa \setminus \gamma \subseteq \beta \subseteq \gamma_2$ and $f_1(s) = f_2(s) = x$ for all $s \in \beta$. Thus $(f_1(s), f_2(s)) \in U$ for all $s \in \beta$. By Assumption (12), there exist $p_1, p_2 \in \mathcal{F}_B$, supported by $\gamma_1 \setminus (\kappa \setminus \gamma)$, such that $(f_1 + p_1, f_2 + p_2) \in T$. Since $\gamma_1 \setminus (\kappa \setminus \gamma) = (\gamma_1 \setminus \kappa) \cup (\gamma_1 \cap \gamma)$, it follows from Corollary 4.4 that $\gamma_1 \setminus (\kappa \setminus \gamma)$ is V -small with respect to (τ_ℓ, B) . Hence,

$$\begin{aligned} & (\tau_\ell(\kappa, x), \tau_\ell(\kappa \setminus \gamma, x)) \\ &= (\tau_\ell(\kappa, x), \ell(f_1)) \circ (\ell(f_1), \ell(f_1 + p)) \circ (\ell(f_1 + p_1), \ell(f_2 + p_2)) \circ \\ & \quad (\ell(f_2 + p_2), \ell(f_2)) \circ (\ell(f_2), \tau_\ell(\kappa \setminus \gamma, x)) \\ & \in V_0^5 \subseteq W. \end{aligned}$$

- .2 There exist $V_0, V_1 \in \text{unif } Z$, $T \in \mathcal{T}$, and $U \in \text{unif } X$, such that (i) $V_0^5 \subseteq W$, (ii) for all $f, g \in \mathcal{F}_B$, if $(f, g) \in T$ then $(\ell(f), \ell(g)) \in V_0$, and (iii) $T \text{ ext}_B U$. Let K be a finite, disjoint subfamily of \mathcal{K} , and let x, y be functions on K to B such that $(x_\kappa, y_\kappa) \in U$ for all $\kappa \in K$. By Proposition 4.2.2, there exists $\gamma \in \mathcal{G}$ such that

- (1) $\bigcup K \subseteq \gamma$,
- (2) $\gamma \setminus \bigcup K \in S(V_1, B, \ell)$.

There exists also G on K to \mathcal{G} with disjoint range, and g, h on K to \mathcal{F} , such that, for all $\kappa \in K$,

- (3) $\kappa \subseteq G_\kappa \subseteq \gamma$,
- (4) $g_\kappa \in \mathcal{F}^*(x_\kappa, \kappa, G_\kappa, B)$, $h_\kappa \in \mathcal{F}^*(y_\kappa, \kappa, G_\kappa, B)$,
- (5) $(\sum_{\kappa \in K} \ell(g_\kappa), \sum_{\kappa \in K} \tau_\ell(\kappa, x_\kappa)) \in V_0$, and $(\sum_{\kappa \in K} \ell(h_\kappa), \sum_{\kappa \in K} \tau_\ell(\kappa, y_\kappa)) \in V_0$.

There exist $p \in \mathcal{F}_*(\bigcup K, \bigcup G, B)$ and $q \in \mathcal{F}_*(\bigcup K, \bigcup G, B)$ such that

$$(\sum_{\kappa \in K} g_\kappa + p, \sum_{\kappa \in K} h_\kappa + q) \in T.$$

Then, by the \mathcal{K} -additivity of ℓ ,

$$\begin{aligned} & (\sum_{\kappa \in K} \tau_\ell(\kappa, x_\kappa), \sum_{\kappa \in K} \tau_\ell(\kappa, y_\kappa)) \\ &= (\sum_{\kappa \in K} \tau_\ell(\kappa, x_\kappa), \sum_{\kappa \in K} \ell(g_\kappa)) \circ (\ell(\sum_{\kappa \in K} g_\kappa), \ell(\sum_{\kappa \in K} g_\kappa + p)) \circ \\ & \quad (\ell(\sum_{\kappa \in K} g_\kappa + p), \ell(\sum_{\kappa \in K} h_\kappa + q)) \circ (\ell(\sum_{\kappa \in K} h_\kappa + q), \ell(\sum_{\kappa \in K} h_\kappa)) \circ \\ & \quad (\sum_{\kappa \in K} \ell(h_\kappa), \sum_{\kappa \in K} \tau_\ell(\kappa, y_\kappa)) \\ & \in V_0^5 \subseteq W. \end{aligned}$$

- .3 Now let $B \in \mathcal{B}$, κ be a disjoint sequence in \mathcal{K} , and $W \in \text{unif } Z$. Choose sequences P, V in $\text{unif } Z$ such that $P_0 \circ P_0 \subseteq W$, $V_0^2 \subseteq P_0$, and $P_{j+1} \circ P_{j+1} \subseteq V_j \subseteq V_j^2 \subseteq P_j$ for all $j \in N$, and a sequence C in \mathcal{B} such that $C_n + C_n \subseteq C_{n+1}$. By Proposition 4.7.3, for all $Q \in \text{unif } Z$, if $\beta_1, \beta_2 \in \mathcal{G}$ are such that $\beta_1 \in S(Q, C_{j+1}, \ell)$ and $\beta_2 \in S(V_{j+1}, C_{j+1}, \ell)$, then $\beta_1 \cup \beta_2$ is $Q \circ P_j \in S(Q \circ P_j, C_j, \ell)$

We shall now construct sequences η, α in \mathcal{K} and γ, β in \mathcal{G} as follows. Let $\eta_0 = \kappa_0$. By Proposition 4.2.2, there exists $\gamma_0 \in \mathcal{G}^+(\eta_0)$ such that $\gamma_0 \setminus \eta_0 \in S(V_0, C_0, \ell)$. By the remark following Assumption (4) there exists $\alpha_0 \in \mathcal{K}$, $\beta_0 \in \mathcal{G}$ such that $\eta_0 \subseteq \beta_0 \subseteq \alpha_0 \subseteq \gamma_0$. Let $\eta_1 = \kappa_1 \setminus \gamma_0$. Since $\eta_{n+1}, \bigcup_{j \leq n} \alpha_j$ are disjoint elements of \mathcal{K} , then, by the remark following Assumption (4), and Proposition 4.2.2, there exists $\gamma_{n+1} \in \mathcal{G}^+(\eta_{n+1})$ such

that $\gamma_{n+1} \cap \bigcup_{j \leq n} \alpha_j = \emptyset$, and $\gamma_{n+1} \setminus \eta_{n+1} \in S(V_{n+1}, C_{n+1}, \ell)$. Now choose $\alpha_{n+1} \in \mathcal{K}$, $\beta_{n+1} \in \mathcal{G}$ with $\eta_{n+1} \subseteq \beta_{n+1} \subseteq \alpha_{n+1} \subseteq \gamma_{n+1}$. We observe that

- (1) β is a disjoint sequence in \mathcal{G} ,
- (2) for each $n \in \mathbf{N}$, $\eta_{n+1} = \kappa_{n+1} \setminus \bigcup_{j \leq n} (\gamma_j \setminus \eta_j)$ and therefore
$$\kappa_{n+1} \subseteq \beta_{n+1} \cup \bigcup_{j \leq n} (\gamma_j \setminus \eta_j).$$

By repeated application of Proposition 4.7.3, $\bigcup_{j \leq n} (\gamma_j \setminus \eta_j) \in S(V_0, C_0, \ell)$. Since ℓ is s -bounded, there exists $m \in \mathbf{N}$ such that $\beta_{n+1} \in S(P_0, C_0, \ell)$ for all $n > m$. Thus, again by Proposition 4.7.3, $\beta_{n+1} \cup \bigcup_{j \leq \mathbf{N}} (\gamma_j \setminus \eta_j) \in S(V_0, B, \ell)$ for all $n > m$. The theorem follows. \square

- .4 By Propositions 4.7.3 and 4.3. \square
- .5 By Propositions 4.2.2, 4.3 and 4.7.1. \square
- .6 Similarly, by Propositions 4.2.1, 4.3 and 4.7.1. \square
- .7 Denote \mathcal{B} -hull(x) by $H_x^{\mathcal{B}}$. Let κ_1 and κ_2 be disjoint members of \mathcal{K} , and $x \in X$. There exists disjoint $\gamma_1, \gamma_2 \in \mathcal{G}$ with $\kappa_i \subseteq \gamma_i$, $i = 1, 2$. For $i = 1, 2$ let $L_i = \{\alpha \in \mathcal{G} : \kappa_i \subseteq \alpha \subseteq \gamma_i\}$, and $M = \{\alpha \in \mathcal{G} : \kappa_1 \cup \kappa_2 \subseteq \alpha \subseteq \gamma_1 \cup \gamma_2\}$, both directed by $\beta \prec \beta'$ iff $\beta' \subseteq \beta$. Since $M = \{\alpha_1 \cup \alpha_2 : \alpha_i \in L_i, i = 1, 2\}$, then

$$\begin{aligned} & \tau_\ell(\kappa_1 \cup \kappa_2, x) \\ &= \lim(\ell^\wedge \mathcal{F}^*(x, \kappa_1 \cup \kappa_2, \gamma, H_x^{\mathcal{B}}) : \gamma \in M) \\ &= \lim(\ell^\wedge (\mathcal{F}^*(x, \kappa_1, \gamma \cap \gamma_1, H_x^{\mathcal{B}}) + \mathcal{F}^*(x, \kappa_2, \gamma \cap \gamma_2, H_x^{\mathcal{B}})) : \gamma \in M) \\ &= \lim(\ell^\wedge \mathcal{F}^*(x, \kappa_1, \gamma \cap \gamma_1, H_x^{\mathcal{B}}) + \ell^\wedge \mathcal{F}^*(x, \kappa_2, \gamma \cap \gamma_2, H_x^{\mathcal{B}}) : \gamma \in M) \\ &= \lim(\ell^\wedge \mathcal{F}^*(x, \kappa_1, \alpha, H_x^{\mathcal{B}}) : \alpha \in L_1) + \lim(\ell^\wedge \mathcal{F}^*(x, \kappa_2, \alpha, H_x^{\mathcal{B}}) : \alpha \in L_2) \\ &= \tau_\ell(\kappa_1, x) + \tau_\ell(\kappa_2, x). \end{aligned} \quad \square$$

Propositions 4.9 *Let \mathcal{R} denote the family of subsets α of S with the following property: for all $B \in \mathcal{B}$ and $V \in \text{unif } Z$, there exist $\kappa \in \mathcal{K}$ and $\gamma \in \mathcal{G}$ such that $\kappa \subseteq \alpha \subseteq \gamma$, and $\gamma \setminus \kappa$ is V -small with respect to (τ_ℓ, B) . Then,*

- .1 $\mathcal{K} \cup \mathcal{G} \subseteq \mathcal{R}$.
- .2 \mathcal{R} is a field.
- .3 $\nu_\ell = \tau_\ell$ on \mathcal{K} .
- .4 For all $B \in \mathcal{B}$ and $\rho \in \mathcal{R}$:
$$\nu_\ell(\rho, x) = \lim (\nu_\ell(\kappa, x) : \kappa \in \mathcal{K}^-(\rho)) = \lim (\nu_\ell(\gamma, x) : \gamma \in \mathcal{G}^+(\rho)),$$
uniformly for $x \in B$.
- .5 ν_ℓ is additive on \mathcal{R} .
- .6 For all $B \in \mathcal{B}$, disjoint sequence ρ in \mathcal{R} and $V \in \text{unif } Z$, there exists $m \in \mathbf{N}$ such that ρ_n is V -small with respect to (ν_ℓ, B) for all $n > m$.
- .7 For all $\alpha \subseteq S$ and $B \in \mathcal{B}$, $\nu_\ell(\alpha, x)$ is defined for all $x \in X$, and $\nu_\ell(\alpha, x) = \lim (\nu_\ell(\gamma, x) : \gamma \in \mathcal{G}^+(\alpha))$, uniformly for $x \in B$.
- .8 For all $B \in \mathcal{B}$ and $V \in \text{unif } Z$, there exists $U \in \text{unif } X$ such that for all finite disjoint $R \subseteq \mathcal{R}$ and functions x, y on R to B , if $(x_\rho, y_\rho) \in U$ for all $\rho \in R$ then $(\sum_{\rho \in R} x_\rho \cdot \nu_\ell(\rho), \sum_{\rho \in R} y_\rho \cdot \nu_\ell(\rho)) \in V$.

Proofs. 4.9

- .1 By Propositions 4.2.1, 4.2.2 and 4.3. \square

- .2 Let $\rho_1, \rho_2 \in \mathcal{R}, V \in \text{unif } Z$ and $B \in \mathcal{B}$. There exists $W \in \text{unif } Z$ such that $W^3 \subseteq V$, and $C \in \mathcal{B}$ with $B + B \subseteq C$. There exist $\kappa_1, \kappa_2 \in \mathcal{K}$ and $\gamma_1, \gamma_2 \in \mathcal{G}$ such that, for $i = 1, 2$, $\kappa_i \subseteq \rho_i \subseteq \gamma_i$, and $\gamma_i \setminus \kappa_i$ is W -small with respect to (τ_ℓ, C) . Now,

$$\begin{aligned} (\gamma_1 \setminus \kappa_2) \setminus (\kappa_1 \setminus \gamma_2) &\subseteq (\gamma_1 \setminus \kappa_1) \cup (\gamma_2 \setminus \kappa_2), \\ (\gamma_1 \cup \gamma_2) \setminus (\kappa_1 \cup \kappa_2) &\subseteq (\gamma_1 \setminus \kappa_1) \cup (\gamma_2 \setminus \kappa_2), \\ \kappa_1 \setminus \gamma_2 &\subseteq \rho_1 \setminus \rho_2 \subseteq \gamma_1 \setminus \kappa_2, \\ \kappa_1 \cup \kappa_2 &\subseteq \rho_1 \cup \rho_2 \subseteq \gamma_1 \cup \gamma_2. \end{aligned}$$

By Proposition 4.3 and Corollary 4.4, the set $(\gamma_1 \setminus \kappa_1) \cup (\gamma_2 \setminus \kappa_2)$ is V -small with respect to (τ_ℓ, B) . Thus \mathcal{R} is a ring, in fact a field, since $S \in \mathcal{G}$, by Assumptions (2) and (10). \square

- .3 By Proposition 4.7.5. \square
.4 By Proposition 4.7.2, and the definitions of \mathcal{R} and ν_ℓ . \square
.5 By Proposition 4.8.3. and the additivity of τ_ℓ . \square
.6 By the definition of \mathcal{R} , and Propositions 4.7.3, 4.7.4. \square
.7 If $\mathcal{G}^+(\alpha)$ has a smallest element the conclusion holds trivially. Otherwise, for each $\gamma \in \mathcal{G}^+(\alpha)$ there exists $\gamma' \in \mathcal{G}^+(\alpha)$ with $\gamma' \subseteq \gamma, \gamma' \neq \gamma$. Suppose then that the net $(\nu_\ell(\gamma, x) : \gamma \in \mathcal{G}^+(\alpha))$ is not Cauchy uniformly for $x \in B$. Then there exists $V \in \text{unif } Z$ such that, for each $\gamma \in \mathcal{G}^+(\alpha)$, there exist $\gamma_1, \gamma_2 \in \mathcal{G}$ with $\gamma_i \subseteq \gamma$ for $i = 1, 2$, and an $x \in B$ such that $(\nu_\ell(\gamma_1, x), \nu_\ell(\gamma_2, x)) \notin V$. Choose $W \in \text{unif } Z$ with $W \circ W \subseteq V$. By the additivity of ν_ℓ on \mathcal{R} ,

$$\begin{aligned} \nu_\ell(\gamma_1, x) &= \nu_\ell(\gamma_1 \setminus \gamma_2, x) + \nu_\ell(\gamma_1 \cap \gamma_2, x), \text{ and} \\ \nu_\ell(\gamma_2, x) &= \nu_\ell(\gamma_2 \setminus \gamma_1, x) + \nu_\ell(\gamma_1 \cap \gamma_2, x), \end{aligned}$$

Since W is translation invariant, then we have either $(0, \nu_\ell(\gamma_1 \setminus \gamma_2, x)) \notin W$ or $(0, \nu_\ell(\gamma_2 \setminus \gamma_1, x)) \notin W$. We may thus construct a disjoint sequence η in \mathcal{R} , and a sequence x in B , such that $(0, \nu_\ell(\eta_n, x_n)) \notin W$. This contradicts Proposition 4.8.6 above. \square

- .8 By Propositions 4.7.2, 4.8.3 and 4.8.4. \square

Definitions 4.10 Let h be a Z^X -valued function on the subsets of S .

$\alpha \subseteq S$ is **regular with respect to** $(h, \mathcal{K}, \mathcal{G}, \mathcal{B})$ if and only if for all $B \in \mathcal{B}$ and $V \in \text{unif } Z$, there exists $\kappa \in \mathcal{K}$, $\gamma \in \mathcal{G}$ such that $\kappa \subseteq \alpha \subseteq \gamma$, and $\gamma \setminus \kappa$ is V -small with respect to $(h|_{\mathcal{K}}, B)$.

Denote by \mathcal{R}_h the set of all $\alpha \subseteq S$ such that α is regular with respect to $(h, \mathcal{K}, \mathcal{G}, \mathcal{B})$. For all $B \in \mathcal{B}$, let $\mathcal{V}(h, \mathcal{K}, \mathcal{G}, B)$ denote the set of all finite sums of the form $\sum_{\rho \in R} x_\rho \cdot h(\rho)$, where R is a finite, disjoint subset of \mathcal{R}_h , and x is a function on R to B .

h is **uniformly partition continuous with respect to** $(\mathcal{K}, \mathcal{G}, \mathcal{B})$ if and only if, for each $W \in \text{unif } Z$ and $B \in \mathcal{B}$, there exists $U \in \text{unif } X$ such that, for all finite, disjoint $R \subseteq \mathcal{R}_h$, and functions x, y on R to B , if $(x_\rho, y_\rho) \in U$ for all $\rho \in R$, then

$$\left(\sum_{\rho \in R} x_\rho \cdot h(\rho), \sum_{\rho \in R} y_\rho \cdot h(\rho) \right) \in W.$$

h is **s-bounded with respect to** $(\mathcal{K}, \mathcal{G}, \mathcal{B})$ if and only if, for each $B \in \mathcal{B}$, disjoint sequence α in \mathcal{R}_h and $V \in \text{unif } Z$, there exists $m \in \mathbf{N}$ such that α_n is V -small with respect to (h, B) for all $n > m$.

We come now to the principal definition and theorem of this section.

Definition 4.11 A Z^X -valued function h on the subsets of S is a **Riesz measure** over $(\mathcal{K}, \mathcal{G}, \mathcal{B})$ if and only if

- (1) $\mathcal{G} \subseteq \mathcal{R}_h$,
- (2) \mathcal{R}_h is a field on which h is additive,
- (3) $h(\gamma) = \lim (h(\kappa) : \kappa \in \mathcal{K}^-(\gamma))$, and $h(\alpha) = \lim (h(\gamma') : \gamma' \in \mathcal{G}^+(\alpha))$, for all $\gamma \in \mathcal{G}$ and $\alpha \subseteq S$,
- (4) h is uniformly partition continuous with respect to $(\mathcal{K}, \mathcal{G}, \mathcal{B})$,
- (5) h is s-bounded with respect to $(\mathcal{K}, \mathcal{G}, \mathcal{B})$.
- (6) for each $B \in \mathcal{B}$, $\mathcal{V}(h, \mathcal{K}, \mathcal{G}, B)$ is a relatively complete subset of Z .

Note that, when Z is a topological vector space, a Riesz measure h over $(\mathcal{K}, \mathcal{G}, \mathcal{B})$ is necessarily a \mathcal{G} -outer measure [?], which is σ -additive on \mathcal{R}_h whenever \mathcal{K} is contained in the family of compact subsets of S , [?], Theorem 1.6. As a consequence of Propositions 4.9, we have

Theorem 4.12 Let ℓ be a Z -valued integral over a Riesz system \mathfrak{R} . Then ν_ℓ is a Riesz measure over $(\mathcal{K}, \mathcal{G}, \mathcal{B})$.

If ℓ is additive then $\text{rng } \nu_\ell$ consists of additive maps. If X and Z are topological vector spaces and ℓ is linear, then $\text{rng } \nu_\ell$ consists of linear maps. By Propositions 4.8.2 and 4.9.4, the range of ν_ℓ is always uniformly continuous on each $B \in \mathcal{B}$. Note also that a linear map on the space $\mathcal{C}_c(\Omega, \mathbf{R}^n)$ of infinitely differentiable functions has an integral representation if it is continuous with respect to the topology of uniform convergence on Ω .

5 Integral Representation

We use an integration process which is based on finite partitions. Conditions under which this produces the same integral as that generated by other processes [?, ?] are given by [?], Theorem 2.5, p. 28.

Let \mathfrak{R} be a Riesz system $(S, (\mathcal{K}, \mathcal{G}), (X, \mathcal{B}), (\mathcal{F}, \mathcal{T}))$, \mathcal{T}_u the uniformity on \mathcal{F} of uniform convergence on S , and μ a Riesz measure over $(\mathcal{K}, \mathcal{G}, \mathcal{B})$. We denote by \mathcal{R}_μ the field of subsets of S which are regular with respect to $(\mu, \mathcal{K}, \mathcal{G}, \mathcal{B})$ (Definition 4.1). For each $E \in \mathcal{R}_\mu$, let $\mathcal{P}_\mu(E)$ denote the collection of all finite, disjoint subfamilies R of \mathcal{R}_μ with $E = \bigcup R$, directed by refinement. For any such finite, disjoint $R \subseteq \mathcal{R}_\mu$, a **choice function** s on R is a function on R such that $s_\rho \in \rho$ for each $\rho \in R$. If f is any function on S to X , we say that f is *integrable over E with respect to μ* if and only if there exists $z \in Z$ such that for all neighbourhoods V of z we can find a finite disjoint $Q \subseteq \mathcal{R}_\mu$, with $E = \bigcup Q$, such that $\sum_{\rho \in R} f(s_\rho) \cdot \mu(\rho) \in V$, whenever s is a choice function on a partition

R of E finer than Q . The limit point $z \in Z$ will be denoted by $\int_E f.d\mu$. We come now to our main result.

Theorem 5.1 *Let ℓ be a Z -valued function on \mathcal{F} . Then ℓ is a Z -valued integral over \mathfrak{R} if and only if, for some unique Riesz measure μ over $(\mathcal{K}, \mathcal{G}, \mathcal{B})$,*

$$\ell(f) = \int_S f.d\mu \text{ for all } f \in \mathcal{F}.$$

The proof is by the following propositions.

Propositions 5.2 *Let \mathfrak{R} be a Riesz system $(S, (\mathcal{K}, \mathcal{G}), (X, \mathcal{B}), (\mathcal{F}, \mathcal{T}))$.*

- .1 *If μ is a Riesz measure over $(\mathcal{K}, \mathcal{G}, \mathcal{B})$, then $\int_E f.d\mu$ is defined for all $f \in \mathcal{F}$ and $E \in \mathcal{R}_\mu$, and $f \in \mathcal{F} \rightarrow \int_S f.d\mu$ is a Z -valued integral over \mathfrak{R} .*
- .2 *If μ_1, μ_2 are Riesz measures over $(\mathcal{K}, \mathcal{G}, \mathcal{B})$ such that $\int_S f.d\mu_1 = \int_S f.d\mu_2$ for all $f \in \mathcal{F}_0$, then $\mu_1 = \mu_2$.*
- .3 *If ℓ is a Z -valued integral over \mathfrak{R} , then ν_ℓ is a Riesz measure over $(\mathcal{K}, \mathcal{G}, \mathcal{B})$, and $\ell(f) = \int_S f.d\nu_\ell$ for all $f \in \mathcal{F}$.*

Proofs.

- .1 Let $f \in \mathcal{F}$. We show first that the sets

$$\left\{ \sum_{\rho \in R} f(s_\rho) \cdot \mu(\rho) : s \text{ is a choice function on } R, \right. \\ \left. \text{and } R \text{ is a finite, disjoint subset of } \mathcal{R}_\mu, \text{ finer than } Q \right\},$$

where Q is a finite, disjoint subfamily of \mathcal{R}_μ with $E = \bigcup Q$, constitute a Cauchy filter base in Z . Since $\mathcal{V}(\mu, B, \mathcal{K}, G)$ is relatively complete (Definition 4.10.6), then this filter base converges to some point of Z .

Let $V \in \text{unif } Z$. Since μ is quasi-uniformly continuous with respect to $(\mathcal{K}, \mathcal{G}, \mathcal{B})$ (Definition 4.10.4), there exists $U \in X$ such that, for all finite disjoint $R \subseteq \mathcal{R}_\mu$ and functions x, y on R to B , if $(x_\rho, y_\rho) \in U$ for all $\rho \in R$, then

$$\left(\sum_{\rho \in R} x_\rho \cdot h(\rho), \sum_{\rho \in R} y_\rho \cdot h(\rho) \right) \in V.$$

Since f is finitely \mathcal{G} -partitionable and $\mathcal{G} \subseteq \mathcal{R}_\mu$ (Definition 4.10.1), there exists a finite disjoint $R_0 \subseteq \mathcal{R}_\mu$ such that $S = \bigcup R_0$ and $(f(s), f(t)) \in U$ for all $\rho \in R$ and $s, t \in \rho$. Let R_1, R_2 be any finite partitions of E by \mathcal{R}_μ which are finer than $\{\rho \cap E : \rho \in R_0\}$, and, for each $i = 1, 2$, let s_i be a choice function on R_i . Now let

$$Q = \{\rho_1 \cap \rho_2 : \rho_1 \in R_1, \rho_2 \in R_2\}$$

and define functions p_1, p_2 on Q by $p_i^\alpha = s_i^\rho$ if $\alpha \subseteq \rho \in R_i$, for $i = 1, 2$. Then, by additivity of μ on \mathcal{R}_μ ,

$$\left(\sum_{\rho \in R_1} f(s_1^\rho) \cdot \mu(\rho), \sum_{\rho \in R_2} f(s_2^\rho) \cdot \mu(\rho) \right) \\ = \left(\sum_{\alpha \in Q} f(p_1^\alpha) \cdot \mu(\alpha), \sum_{\alpha \in Q} f(p_2^\alpha) \cdot \mu(\alpha) \right) \in V,$$

since $(f(p_1^\alpha), f(p_2^\alpha)) \in U$ for all $\alpha \in Q$. Let $\int_E f.d\mu$ be the limit to which the filter base converges. It is easily checked that the map $f \rightarrow \int_S f.d\mu$ is a Z -valued integral over \mathfrak{R} . \square

.2 Certainly, if μ is a Riesz measure over $(\mathcal{K}, \mathcal{G}, \mathcal{B})$ then, for each $f \in \mathcal{F}$, the set function $\rho \in \mathcal{R}_\mu \rightarrow \int_\rho f.d\mu$ is additive [?]. Let $\kappa \in \mathcal{K}, x \in X$, and $V \in \text{unif } Z$. Choose $W \in \text{unif } Z$ such that $W \circ W \subseteq V$. Since $\kappa \in \mathcal{R}(\mu_i, \mathcal{K}, \mathcal{G}, \mathcal{B})$ for $i = 1, 2$, there exists $\gamma \in \mathcal{G}$ with $\kappa \subseteq \gamma$ such that $\gamma \setminus \kappa$ is W -small with respect to (μ_i, B) . Then $\int_{\gamma \setminus \kappa} f.d\mu_i \in W$ for all $f \in \mathcal{F}_B$ with $f \prec \gamma \setminus \kappa$. By Assumption (11), choose $f \in \mathcal{F}_0$ such that $\kappa =_x f \prec \gamma$. Then

$$\begin{aligned} & (x.\mu_1(\kappa), x.\mu_2(\kappa)) \\ &= (\int_\kappa f.d\mu_1, \int_\kappa f.d\mu_2) \\ &= (\int_\kappa f.d\mu_1, \int_\kappa f.d\mu_1 + \int_{\gamma \setminus \kappa} f.d\mu_1) \circ (\int_\kappa f.d\mu_2 + \int_{\gamma \setminus \kappa} f.d\mu_2, \int_\kappa f.d\mu_2) \\ &\varepsilon W \circ W, \text{ by translation invariance of } W, \\ &\subseteq V \end{aligned}$$

Thus μ_1, μ_2 agree on \mathcal{K} , and therefore on all subsets of S . \square

.3 Let $B, C \in \mathcal{B}$ be such that $B + B \subseteq C$, and $f \in \mathcal{F}_B$. We can find a finite sum $\sum_{\rho \in R} f(s_\rho).\nu_\ell(\rho)$ which is arbitrarily close to $\int_S f.d\nu_\ell$, and a function $g \in \mathcal{F}$, arbitrarily close to f , for which $\ell(g)$ is arbitrarily close to the finite sum. It follows that $\int_S f.d\nu_\ell = \ell(f)$, since ℓ is uniformly continuous. The details of the proof are given below.

Let $V \in \text{unif } Z$. There exists $W \in \text{unif } Z$, $T \in \mathcal{T}$ and $U \in \text{unif } X$ such that

- (i) $W^6 \subseteq V$,
- (ii) if $f_1, f_2 \in \mathcal{F}_C$, and $(f_1, f_2) \in T$ then $(\ell(f_1), \ell(f_2)) \in W$,
- (iii) $T \text{ ext}_B U$.

Since f is finitely \mathcal{G} -partitionable there exists $n \in \mathbb{N}$ and $\{G_0, \dots, G_{n-1}\} \subseteq \mathcal{G}$ such that $S = \bigcup_{i \leq n-1} G_i$, and $(f(s), f(t)) \in U$ for each $i \leq n-1$ and all $s, t \in G_i$. Let $\bar{R}_i = G_i \setminus \bigcup_{j < i} G_j$, and $s_i \in R_i$ for each $i \leq n-1$ (without loss of generality, we may assume that $R_i \neq \emptyset$ for all $i \leq n-1$). Then $R_i \subseteq G_i$ and, as in the proof of 4.1 above,

$$(\int_S f.d\nu_\ell, \sum_{i \leq n-1} f(s_i).\nu_\ell(R_i)) \in V.$$

Since R_i is regular with respect to $(\nu_\ell, \mathcal{K}, \mathcal{G}, \mathcal{B})$, choose disjoint $\kappa_i \in \mathcal{K}$, $i \leq n-1$, such that $\kappa_i \subseteq R_i$, $S \setminus \bigcup_{i \leq n-1} \kappa_i \in S(W, B, \ell)$, and

$$(\sum_{i \leq n-1} f(s_i).\nu_\ell(R_i), \sum_{i \leq n-1} \tau_\ell(\kappa_i, f(s_i))) \in W$$

Choose now $\beta \in \mathcal{G}$, disjoint $\{P_0, \dots, P_{n-1}\} \subseteq \mathcal{G}$, $g_i \in \mathcal{F}^*(f(s_i), \kappa_i, P_i, B)$ and $\delta_i \in \mathcal{G}$, $i \leq n-1$, such that

- (i) $\bigcup_{i \leq n-1} \kappa_i \subseteq \beta$
- (ii) $\kappa_i \subseteq \delta_i \subseteq P_i \subseteq \beta$, $i \leq n-1$,
- (iii) $(\sum_{i \leq n-1} \tau_\ell(\kappa_i, f(s_i)), \sum_{i \leq n-1} \ell(g_i)) \in W_0$,
- (iv) $g_i(t) = f(s_i)$ for all $t \in \delta_i$, $i \leq n-1$.

Let $h = \sum_{i \leq n-1} g_i$ and $\omega = \bigcup_{i \leq n-1} \delta_i$, then $(h(t), f(t)) \in U$ for all $t \in \omega$. Hence, there exist $p, g \in \mathcal{F}_B$, both supported by $S \setminus \bigcup_{i \leq n-1} \kappa_i$, such that $(h + q, f + p) \in T$. Then

$$\begin{aligned}
& (\int_S f.d \nu_\ell, \ell(f)) \\
&= (\int_S f.d \nu_\ell, \sum_{i \leq n-1} f(s_i) \cdot \nu_\ell(R_i)) \circ \\
&\quad (\sum_{i \leq n-1} f(s_i) \cdot \nu_\ell(R_i), \sum_{i \leq n-1} \tau_\ell(\kappa_i, f(s_i))) \circ \\
&\quad (\sum_{i \leq n-1} \tau_\ell(\kappa_i, f(s_i)), \ell(h)) \circ \\
&\quad (\ell(h), \ell(h+q)) \circ (\ell(h+q), \ell(f+p)) \circ (\ell(f+p), \ell(f)) \\
&\in W^6 \subseteq V. \\
&\text{Hence } \int_S f.d \nu_\ell = \ell(f). \quad \square
\end{aligned}$$

The theorem follows by Propositions 5.2.1, 5.2.2 and 5.2.3. Taking these together with Example 3.6.6 and the comments of the Introduction, we deduce the following assertion.

Theorem 5.3 *Let \mathfrak{R} be a Riesz system $(S, (\mathcal{K}, \mathcal{G}), (X, \mathcal{B}), (\mathcal{F}, T))$, with X being a topological vector space, and $\mathcal{F} \subseteq \mathcal{C}_\mathcal{K}(S, X)$. Let Z be a topological vector space in which every bounded subset is relatively complete and perfect, and ℓ be a map on \mathcal{F} into Z which is uniformly continuous on \mathcal{F}_B , and maps it into a bounded subset of Z , for all $B \in \mathcal{B}$. Then, ℓ has the Hammerstein property relative to \mathcal{K} if and only if there exists a unique Riesz measure μ over $(\mathcal{K}, \mathcal{G}, \mathcal{B})$, with values in $L_\mathcal{B}(X, Z)$, such that $\ell(f) = \int_S f.d\mu$ for all $f \in \mathcal{F}$.*

Applying the characterization of integrals given by theorem 3.4 to Propositions 5.2 we have

Theorem 5.4 *Let \mathfrak{R} be a Riesz system $(S, (\mathcal{K}, \mathcal{G}), (X, \mathcal{B}), (\mathcal{F}, T))$, with X being a topological vector space, and $\mathcal{F} \subseteq \mathcal{C}_\mathcal{K}(S, X)$. Let Z be a topological vector space in which every bounded subset is relatively complete and perfect. Then, ℓ is a uniformly continuous, linear map on \mathcal{F} to Z if and only if there exists a unique $L_\mathcal{B}(X, Z)$ -valued Riesz measure μ over $(\mathcal{K}, \mathcal{G}, \mathcal{B})$ such that $\ell(f) = \int_S f.d\mu$ for all $f \in \mathcal{F}$.*

Proof. By Theorem 3.4, ℓ is an integral over the Riesz system \mathfrak{R} . The result follows by Propositions 5.2.1 – 5.2.3 and the continuity properties of the operator $f \rightarrow \int_S f.d\nu_\ell$. \square

Theorems 5.5 *Let \mathfrak{R} be a Riesz system $(S, (\mathcal{K}, \mathcal{G}), (X, \mathcal{B}), (\mathcal{F}, T))$ in which S is a topological space quasi-normal under $(\mathcal{K}, \mathcal{G})$, X is a topological vector space, \mathcal{B} is a subfamily of the family of closed, balanced, totally bounded subsets of X , $\mathcal{F} \subseteq \mathcal{C}_p(S, X)$ is a linear space of uniformly continuous functions on S to X which is a module over $\mathcal{C}_p(S)$ and contains $X \otimes \mathcal{C}_\mathcal{K}(S)$, and T is a uniformity on \mathcal{F} coarser than that of uniform convergence on S (Remarks 2.5). Let Z be a locally convex space.*

- .1 *If ℓ is a uniformly continuous linear map on \mathcal{F} to Z which maps bounded sets into relatively complete sets, and has partial operators ℓ_x , $x \in X$, which map bounded subsets of $\mathcal{C}_p(X)$ into relatively weakly-compact subsets of Z , then there exists a unique $L_\mathcal{B}(X, Z)$ -valued Riesz measure μ over $(\mathcal{K}, \mathcal{G}, \mathcal{B})$ such that $\ell(f) = \int_S f.d\mu$ for all $f \in \mathcal{F}_0$.*
- .2 *If ℓ is a uniformly continuous linear operator on \mathcal{F} to Z which maps bounded subsets into relatively complete, relatively weakly compact subsets,*

then there exists a unique $L_{\mathcal{B}}(X, Z)$ -valued Riesz measure μ over $(\mathcal{K}, \mathcal{G}, \mathcal{B})$ such that $\ell(f) = \int_S f.d\mu$ for all $f \in \mathcal{F}$.

- .3 If ℓ is any uniformly continuous linear map on \mathcal{F} to Z , then there exists a unique $L_{\mathcal{B}}(X, Z''_{\sigma})$ -valued Riesz measure μ over $(\mathcal{K}, \mathcal{G}, \mathcal{B})$ such that $\ell(f) = \int_S f.d\mu$ for all $f \in \mathcal{F}$.

Proofs.

- .1 By Example 3.6.1, and Propositions 5.2. \square
.2 By Example 3.6.2, and Propositions 5.2. \square
.3 Let $A \subseteq Z$ be bounded. Then A^{00} is the $\sigma(Z'', Z)$ -closed absolutely convex hull of A . Since $A^0 \in nbhd 0$ in $Z'_{\mathcal{B}}$, then A^{00} is $\sigma(Z'', Z')$ -compact ([?], pp.35,61). Identifying Z with a subspace of Z'' in the usual manner, it follows that ℓ maps bounded subsets of \mathcal{F} into relatively $\sigma(Z'', Z')$ -compact subsets of Z'' . The result now follows by Theorem 5.1. \square

Remarks 5.6

- .1 When S is locally compact the theory yields integral representations of linear maps ℓ on spaces $\mathcal{C}(S, X)$ with the uniformity of uniform convergence on compacta [?]. Let S be locally compact, \mathcal{K} its family of closed compact subsets and \mathcal{G} its family of open subsets. Let X be a topological vector space and \mathcal{B} its family of balanced, totally bounded subsets. Let \mathcal{T}_u be the uniformity for $\mathcal{C}_c(S, X)$ of uniform convergence on S , and \mathcal{T}_c the uniformity for $\mathcal{C}(S, X)$ of uniform convergence on compacta. Let Z be a topological vector space such that each bounded subset of Z is relatively complete and perfect. Let ℓ be a \mathcal{T}_c -uniformly continuous, linear operator on $\mathcal{C}(S, X)$ to Z . Now $\mathfrak{R}^u = (S, (\mathcal{K}, \mathcal{G}), (X, \mathcal{B}), (\mathcal{C}_c(S, X), \mathcal{T}_u))$ is a Riesz system (Example 2.4.2), and the restriction of ℓ to $\mathcal{C}_c(S, X)$ is an integral over \mathfrak{R}^u . Thus, by Proposition 5.2.3, $\ell(f) = \int_S f.d\nu_{\ell}$ for all $f \in \mathcal{C}_c(S, X)$. Clearly, $\mathcal{C}_c(S, X)$ is dense in $\mathcal{C}(S, X)$ for the topology of uniform convergence on compacta. Thus $\ell(f) = \int_S f.d\nu_{\ell}$ for all $f \in \mathcal{C}(S, X)$, provided that $f \rightarrow \int_S f.d\nu_{\ell}$ is defined for all $f \in \mathcal{C}(S, X)$, and is uniformly continuous for the topology of uniform convergence on compacta.

These last statements do in fact hold. Let $V \in unif Z$. Then there exists $U \in unif X$ and $K \in \mathcal{K}$ such that, for all $f, g \in \mathcal{C}(S, X)$, if $(f(s), g(s)) \in U$ for all $s \in K$ then $(\ell(f), \ell(g)) \in V$. Thus $\ell(f) \in V$ for all $f \in \mathcal{F}$ supported by $S \setminus K$. By Proposition 4.3 it follows that $\sum_{\rho \in R} x_{\rho} \cdot \nu_{\ell}(\rho) \in V$, for all finite disjoint $R \subseteq \mathcal{R}(\nu_{\ell}, \mathcal{K}, \mathcal{G}, \mathcal{B})$ with $K \cap \bigcup R = \phi$, and choice function $x : \rho \in R \rightarrow x_{\rho} \in \rho$. By a straightforward extension of the proof of Proposition 5.2.1, we can show that $\int_S f.d\nu_{\ell}$ exists for all $f \in \mathcal{C}(S, X)$ and that $f \in \mathcal{C}(S, X) \rightarrow \int_S f.d\nu_{\ell}$ is uniformly continuous for the topology of uniform convergence on compacta.

- .2 More generally, let \mathfrak{R} be a Riesz system $(S, (\mathcal{K}, \mathcal{G}), (X, \mathcal{B}), (\mathcal{F}, T))$ in which S is a topological space quasi-normal under $(\mathcal{K}, \mathcal{G})$, X is a topological vector space, \mathcal{B} is a subfamily of the family of closed, balanced, totally bounded subsets of X , $\mathcal{F} \subseteq \mathcal{C}_p(S, X)$ is a space of uniformly continuous

functions on S to X which is a module over $\mathcal{C}_p(S)$ and contains $X \otimes \mathcal{C}_K(S)$, and \mathcal{T} is the uniformity on \mathcal{F} of uniform convergence on the members of some $\mathcal{D} \subseteq \mathcal{K}$. As above, we can now check that ℓ is a Z -valued integral over \mathfrak{R} if and only if there is a unique Riesz measure μ over \mathfrak{R} such that

$$\ell(f) = \int_S f.d\mu, \text{ for all } f \in \mathcal{F},$$

and, for all $W \in \text{unif}Z$ and $B \in \mathcal{B}$, there exists $D \in \mathcal{D}$ such that $S \setminus D$ is W -small with respect to (μ, B) .

- .3 The existence of Riesz measures σ -additive on a σ -field raises no new problems, and can be treated using generalizations of results due to P. Alexandroff and E. Marczewski [?, ?, ?]. Indeed, when S is locally compact, the measure ν_ℓ is σ -additive on a σ -ring containing \mathcal{K} , the family of compact subsets of S (Theorem 1.5 of [?]). A more general condition can be given on ℓ . A subset α of S will be called **U -small** with respect to ℓ iff $(\ell(f), \ell(f+g)) \in U$ for all $g \prec \alpha$. ℓ will be called **\mathcal{G} -bounded** iff for all $U \in \text{unif}Z$, $B \in \mathcal{B}$, $K \in \mathcal{K}$ and $\mathcal{G}' \subseteq \mathcal{G}$ with $K \subseteq \bigcup \mathcal{G}'$, there exists a finite $\mathcal{H} \subseteq \mathcal{G}'$ such that $K \setminus \bigcup \mathcal{H}$ is U -small W -small with respect to ℓ . An extension of Theorem 1.5 of [?] then shows

Theorem 5.7 *Let X and Z be uniform commutative monoids, and \mathfrak{R} be a Riesz system. If ℓ is an integral which is \mathcal{G} -bounded then ν_ℓ is σ -additive.*

Note that a subset α of S is in $S(W, B, \ell)$ if and only if it is W -small with respect to ℓ , since the Riesz measure, ν_ℓ , admits approximation from above by \mathcal{G} .

- .4 For X, Z locally convex and S Hausdorff, locally compact, Theorem 5.4.1 implies the main result of [?], while Theorem 5.4.3 yields the essential content of Theorems 0.1, 6.2 and 6.3 of [?]. Results of [?, ?] may be derived from Theorem 5.4.1 as is done in [?].
- .5 Combining Example 3.6.5 with Proposition 5.2.3 we obtain integral representations for dominated operators. These extend results of [?], p. 380, in that the underlying topological space S may now be normal, and is not restricted to the locally compact case.
- .6 We apply the foregoing discussion to that of [?], showing that uniformly continuous, linear maps on certain families of uniformly continuous, vector-valued functions on infinite-dimensional spaces generate integrals. In keeping with the notation and assumptions of that paper, we shall assume the following:

- (1) $(X, |\cdot|_X)$ and $(Z, |\cdot|_Z)$ are normed spaces, and ω is a strictly decreasing, positive function on \mathbf{R} , with $\omega(t) \rightarrow 0$ as $t \rightarrow \infty$,
- (2) for each positive integer $i \in \mathbf{N}$, $(S_i, |\cdot|_i)$ is a normed space, with $S_i \subseteq S_j$ and $|x|_i > |x|_j$, if $i < j$,
- (3) $S = \bigcup_{i \in \mathbf{N}} S_i$ has the inductive topology [?], and is such that every bounded subset is precompact (that is, the completion of S is a Montel space);

- (4) for any X -valued function g on S_i , $\|g\|_i = \sup_{x \in S_i} |g(x)|_X \omega(|x|_i)$,
- (5) $\mathcal{C}_i(S_i, X)$ is the family of all uniformly continuous X -valued functions f on S_i which are uniformly continuous on bounded subsets of S_i , and $\|f\|_i < \infty$,
- (6) $\mathcal{C}_\infty(S, X)$ is the family of all continuous X -valued functions f on S which are uniformly continuous on each bounded subset of S , and for each positive integer $i \in \mathbf{N}$,

$$\|f\|_{S,i} = \sup_{x \in S} |f(x)|_X \omega(|x|_i) < \infty,$$

- (7) \mathcal{F}_∞ is $\mathcal{C}_\infty(S, X)$ with the uniformity \mathcal{U}_∞ generated by the norms $\|\cdot\|_{S,i}$, and \mathcal{F}_i is $\mathcal{C}_\infty(S, X)$ with the uniformity \mathcal{U}_i generated by the norm $\|\cdot\|_i$.

Now let ℓ be a uniformly continuous linear map from \mathcal{F}_∞ to Z . Then there exist $i \in P$ and $\delta > 0$ such that $|\ell f|_Z \leq 1$ if $\|f\|_{S,i} \leq \delta$, and therefore ℓ induces a linear map ℓ_i from \mathcal{F}_i to Z , which is uniformly continuous on \mathcal{F}_i with respect to the uniformity generated by $\|\cdot\|_i$. Since $(S_i, |\cdot|_i)$ is metrisable, it is normal. Let \mathcal{K}_i be the corresponding family of closed subsets of S_i , \mathcal{G}_i the family of corresponding open subsets of S_i , and \mathcal{B} the family of closed, totally bounded subsets of X . Then $\mathfrak{R} = (S_i, (\mathcal{K}_i, \mathcal{G}_i), (X, \mathcal{B}), (\mathcal{F}_i, \mathcal{U}_i))$ is a Riesz system, by Example 2.4.3 and Remark 2.5.1. If $\iota : Z \rightarrow Z''$ is the canonical embedding, then $\iota \circ \ell$ is an integral over \mathfrak{R} , by Example 3.5.3. Applying the previous theory (Theorem 5.5.1), we have a representation

$$\iota \circ \ell(f) = \int f.d\nu_i$$

for some finitely additive measure, ν_i , on a ring of subsets of S_i containing $\mathcal{K}_i \cup \mathcal{G}_i$.

- .7 Together with the notation of Example 2.4.3, let $Z = L_0(\lambda)$ for some probability measure λ . Then, as a consequence of Remark 2.5.2, Remark 3.5.7 and Theorem 5.4, if S is quasi-normal w.r.t. $(\mathcal{K}, \mathcal{G})$, every uniformly continuous linear map ℓ from $\mathcal{C}_p(S, X)$ to Z is given by integration with respect to a Z^X -valued Riesz measure over $(\mathcal{K}, \mathcal{G}, \mathcal{B})$.
- .8 The preceding theory does not exhibit an homeomorphism between the space of Riesz measures and the space of the corresponding integrals. The construction of such an homeomorphism should be a straightforward generalization of known techniques [?, ?, ?, ?].

Notwithstanding many applications to topological vector spaces, the foregoing theory has been derived for functions, operators, and measures with values in uniform commutative monoids. These arise naturally when one considers set-valued functions [?, ?]. Along with X and Z being uniform commutative monoids, we may also take S to be quasi-normal, in particular, either normal or locally compact (thus S can be any metric space [?, ?]).

It was observed in the introduction that an operator with an integral representation necessarily has the Hammerstein property. For topological vector spaces X, Z and \mathcal{F} , the theory shows that an operator with the Hammerstein property is necessarily an integral, with respect to a given Riesz system, $(S, (\mathcal{K}, \mathcal{G}), (X, \mathcal{B}), (\mathcal{F}, \mathcal{T}))$, and therefore has an integral representation. Thus, when X, Z and \mathcal{F} are topological vector spaces, the present theory yields a bijection between operators with the Hammerstein property and the family of their associated Riesz measures (Example 3.6.6, Theorem 5.1, Theorem 5.3, [?]).

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